We want students to learn to reason mathematically and communicate their reasoning coherently. To reason mathematically means providing a logical argument for any claim that you make, based only on prior knowledge of mathematics. In mathematical parlance, this process of providing a logical argument to justify a claim is known as the “proof of the claim”. At any given point in the course, the prior knowledge consists of the prerequisite mathematics and the mathematics students have learned up to that point. The prerequisite mathematics include mathematics of K-12 and the content of a college algebra course. In the process of learning, we hope that students will acquire skills that are required in calculus and beyond.

I would like to thank Larry Francis and Bob Campbell for editing this document.
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Part I

Algebra
Chapter 1

Equations and Inequalities

1.1 Quadratic Equations

I will assume that you have learned how to factor a polynomial, (if possible), in a prior course. In this section, we want to make sure you know how to solve a quadratic equation by completing the square.

Given any real number \( a \), either \( a > 0 \), or \( a = 0 \), or \( a < 0 \). This is known as the Trichotomy Law of real numbers.

Recall the following claim that you may have learned in high school.

**Zero Product Property Theorem.** If \( A \) and \( B \) are real numbers and \( AB = 0 \), then either \( A = 0 \) or \( B = 0 \).

**Proof.** Assume both \( A \) and \( B \) are not zero. Then by the Trichotomy Law for real numbers, \( A > 0 \) or \( A < 0 \) and \( B > 0 \) or \( B < 0 \). If \( A \) and \( B \) have the same sign, then \( AB > 0 \). But this contradicts the fact that \( AB = 0 \). Therefore, \( A \) and \( B \) cannot have the same sign. If \( A \) and \( B \) have opposite signs, then \( AB < 0 \). This too contradicts that fact that \( AB = 0 \). That is, \( A \) and \( B \) cannot have opposite signs either. That means our assumption must be false. That is, by the Trichotomy Law, if \( AB = 0 \), then either \( A = 0 \) or \( B = 0 \).

\( \square \)

\(^{1}\)Usually, a claim that you can prove is called a theorem.
The following is another theorem that you can prove using the Trichotomy Law for real numbers.

**Squares are Nonnegative Theorem.** If $x$ is a real number, then $x^2 \geq 0$.

**Exercise.** Prove the Squares are Nonnegative Theorem.

You may have learned in high school that we sometimes use English letters to represent numbers. (You have seen this practice in the two previous theorems.) If a letter represents a number that can be chosen from a (finite or infinite) range of numbers, then we say that letter represents a variable. If a letter represents a fixed number, then we say that letter represents a constant. Historically, letters $x, y, z$ are used to represent variables, and letters $a, b, c$ are used to represent constants.

If an equation in $x$ is a true statement for all possible values of $x$, then it is known as an identity in $x$. The following are three important identities that you may have seen in high school.

**Three Identities Theorem.** Suppose $a$ is any positive constant and $x$ is any real number. Then

1. $x^2 - (\sqrt{a})^2 = (x - \sqrt{a})(x + \sqrt{a})$
2. $(x + a)^2 = x^2 + 2ax + a^2$
3. $(x - a)^2 = x^2 - 2ax + a^2$

**Exercise.** Prove the above theorem.

The first identity is known as the Difference of Squares Identity, and the other two are known as Binomial Square Identities.

Usually an equation in $x$ is not a true statement for most – or even for all – values of $x$. An equation in variable $x$ is considered as an invitation to find the values of $x$ that make the given equation a true statement. For example, $x^2 = 4$ is an equation in $x$, where $x$ is a real number. That is, $x$ can be any real number. For example, $\frac{1}{2}$ is a real number. But
\((\frac{1}{2})^2 \neq 4\). Therefore, the equation is not a true statement when \(x = \frac{1}{2}\). In fact, the given equation is not true for a lot of real numbers. In this case, you might be able to guess that \(x\) has to be either 2 or \(-2\) for the given equation to be a true statement. We say 2 and \(-2\) are the solutions of the equation \(x^2 = 4\).

Can you always guess the solutions of an equation in \(x\)? Can you guess the solutions of the equation \(345x^2 - 576x - 3245 = 0\)? It would be nice if we could develop an algebraic method to find solutions of quadratic equations in \(x\).

The following is a theorem that you may have learned in K-12.

**Four Properties Theorem.** Suppose \(A, B\) and \(C\) are real numbers.

1. If \(A = B\), then \(A + C = B + C\).
2. If \(A = B\), then \(A - C = B - C\).
3. If \(A = B\), then \(AC = BC\).
4. If \(A = B\), then \(\frac{A}{C} = \frac{B}{C}\), provided \(C \neq 0\).

It is very important that you understand what this theorem says and what it does not say. Let us look at the first statement carefully. If we know that \(A = B\), then we know that \(A + C = B + C\). That is, if \(A = B\) is a true statement, then we know that \(A + C = B + C\) is also a true statement. For example, we know that \(\frac{14}{7} = 2\). Therefore, by this theorem, we know that \(\frac{14}{7} + \frac{5678}{29867} = 2 + \frac{5678}{29867}\) is also true. But if we do not know that \(A = B\) is true, then none of the conclusions of this theorem may be valid, because \(A = B\) is the requirement for all four statements.

Let us go back to the equation \(x^2 = 4\). We know that this equation is NOT true for almost all real numbers. (By guess-and-check we discovered that \(x^2 = 4\) is true for \(x = 2\) and \(x = -2\).) In the following theorem we will develop a mathematical method that we can use to find solutions of equations in \(x\), in general.

**Theorem.** Prove that the only solutions of the equation \(x^2 = 4\) are 2 and \(-2\).

**Proof.** Let us assume that \(x^2 = 4\) is a true statement for some real number \(x\). In other words, we are assuming that \(x^2 = 4\) is a TRUE statement for some number \(x\). Then by the Four Properties Theorem, \(x^2 - 4 = 4 - 4\). That is, \(x^2 - 4 = 0\) is a true statement. Then
$(x - 2)(x + 2) = 0$ is a true statement, by the Difference of Squares Identity. Then either $x - 2 = 0$ or $x + 2 = 0$, by the Zero Product Property Theorem. By the Four Properties Theorem, either $x = 2$ or $x = -2$. Our whole argument is based on our assumption that $x^2 = 4$ is a true statement for some real number $x$. Therefore, we must check and see if our assumption is in fact correct. If $x = 2$, then $2^2 = 4$ is true. If $x = -2$, then $(-2)^2 = 4$ is also true. Therefore, our assumption is true for both $x = 2$ and $x = -2$. That is, 2 and $-2$ are solutions of $x^2 = 4$.

In general, you can prove the following theorem using a similar argument as in the previous theorem.

**Square-root Principle Theorem.** Suppose $a$ is a positive constant. Then the solutions of the equation $x^2 = a$ are $\sqrt{a}$ and $-\sqrt{a}$.

**Exercise.** Prove the Square-root Principle Theorem.

The next theorem is a stepping stone to finding solutions of more general quadratic equations.

**Theorem.** Prove that the only solutions of the equation $(x - 5)^2 = 4$ are $5 + 2$ and $5 - 2$.

**Proof.** Let us assume that $(x - 5)^2 = 4$ is a true statement for some real number $x$. Let $X = x - 5$. Then our assumption becomes: $X^2 = 4$ is a true statement. Then by Square-root Principle Theorem, $X = 2$ and $X = -2$ are true statements. Substituting back, $x - 5 = 2$ and $x - 5 = -2$ are true statements. This implies that either $x = 5 + 2$ or $x = 5 - 2$, by the Four Properties Theorem. Check and see if these numbers are solutions of the given equation. If $x = 5 + 2$, then $((5 + 2) - 5)^2 = 2^2 = 4$ is true. If $x = 5 - 2$, then $((5 - 2) - 5)^2 = (-2)^2 = 4$ is true. Therefore, our assumption is correct for both $x = 5 + 2$ and $x = 5 - 2$. That is, $5 + 2$ and $5 - 2$ are solutions of $(x - 5)^2 = 4$. □

In general, you can prove the following theorem using a similar argument as in the previous theorem.
Theorem (Theorem 1). Suppose $a$ and $b$ are constants and $a > 0$. Then the solutions of the equation $(x - b)^2 = a$ are $b + \sqrt{a}$ and $b - \sqrt{a}$.

Exercise. Prove Theorem 1.

Take a closer look at the Binomial Square Identities.

Suppose $a$ is any positive constant and $x$ is any real number. Then

1. $(x + a)^2 = x^2 + 2ax + a^2$
2. $(x - a)^2 = x^2 - 2ax + a^2$

In both identities, the leading coefficient of the right side of the identity is 1, and the constant term is the square of the half of the absolute value of the coefficient of $x$. That is, given the first two terms of a binomial square, we can predict the third term of the square. In other words, given the first two terms (with leading coefficient 1), we can complete the square by adding the square of one half of the absolute value of the coefficient of $x$ as the third term.

Theorem. Prove that the only solutions of the equation $x^2 - 4x - 1 = 0$ are $2 + \sqrt{5}$ and $2 - \sqrt{5}$.

Proof. Assume that $x^2 - 4x - 1 = 0$ is a true statement for some real number $x$. Then by the Four Properties Theorem, $x^2 - 4x = 1$ is true. On the left side of this true statement we have two terms. The first term is $x^2$ with coefficient 1, and the second term is $-4x$ with coefficient $-4$. As we observed earlier, we can complete the square of the left side by adding half of the absolute value of $-4$ square to left side. However, this will make the given true statement false. We can use the Four Properties Theorem, and add $(2)^2$ to both sides, and get a new true statement from the existing true statement. Therefore, $x^2 - 4x + (2)^2 = 1 + (2)^2$ is true. By recognizing the square on the left side (according to the second binomial square identity above), $(x - 2)^2 = 5$ is true. It should be clear at this point why we wanted to complete the square. As you can see, by doing so we have converted the given equation in to an equation of the form given in Theorem 1. That is, at this point we can use the full force of Theorem 1. By Theorem 1, $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$. Let us check and see if our assumption is true. If $x = 2 + \sqrt{5}$, then $x^2 - 4x - 1 = (2 + \sqrt{5})^2 - 4(2 + \sqrt{5}) - 1 = (4 + 4\sqrt{5} + 5) - 8 - 4\sqrt{5} - 1 = 0.$
Therefore, our assumption is true when \( x = 2 + \sqrt{5} \). If \( x = 2 - \sqrt{5} \), then \( x^2 - 4x - 1 = (2 - \sqrt{5})^2 - 4(2 - \sqrt{5}) - 1 = (4 - 4\sqrt{5} + 5) - 8 + 4\sqrt{5} - 1 = 0 \). Therefore, our assumption is also true when \( x = 2 - \sqrt{5} \). That is, the solutions of \( x^2 - 4x - 1 = 0 \) are \( 2 + \sqrt{5} \) and \( 2 - \sqrt{5} \).

**Theorem.** Prove that the only solutions of the equation \( 3x^2 - 12x - 3 = 0 \) are \( 2 + \sqrt{5} \) and \( 2 - \sqrt{5} \).

**Proof.** Assume that \( 3x^2 - 12x - 3 = 0 \) is a true statement for some real number \( x \). Then by using the fourth property of the Four Properties Theorem, \( x^2 - 4x - 1 = 0 \) is true. Then by the previous theorem, the result follows. [Do not forget to check your answers with the given equation to see if your assumption is true.]

### 1.2 Rational Equations

Solutions to any rational equation can be found by using the method demonstrated in the following example.

**Theorem.** The only solution of the equation \( \frac{x}{x+1} - \frac{2}{x-1} = \frac{-4}{(x^2 - 1)} \) is 2.

**Proof.** First, we recognize that the given equation is the same as

\[
\frac{x}{x+1} - \frac{2}{x-1} = \frac{-4}{(x+1)(x-1)},
\]

by using the Difference of Squares Identity.

Assume that

\[
\frac{x}{x+1} - \frac{2}{x-1} = \frac{-4}{(x+1)(x-1)}
\]

is true for some real number \( x \).

Then by using the third property of the Four Properties Theorem, we multiply both sides by \( (x-1)(x+1) \). That is,

\[
(x-1)(x+1) \left[ \frac{x}{x+1} - \frac{2}{x-1} \right] = (x-1)(x+1) \left[ \frac{-4}{(x+1)(x-1)} \right],
\]

is true.
By using the distributive property,

\[(x - 1)(x + 1) \frac{x}{(x + 1)} - (x - 1)(x + 1) \frac{2}{(x - 1)} = (x - 1)(x + 1) \left[ \frac{-4}{(x + 1)(x - 1)} \right],\]

is true.

By using the multiplicative inverse property, \((A \cdot \frac{1}{A} = 1, \text{ for any non-zero real number } A)\)

\[1 \cdot x(x - 1) - 1 \cdot 2(x + 1) = 1 \cdot (-4)\]

is true.

By using the multiplicative identity property, \((A \cdot 1 = A, \text{ for any real number } A),\)

\[x(x - 1) - 2(x + 1) = -4\]

is true.

By using the distributive property and collecting like terms, we have

\[x^2 - 3x - 2 = -4\]

is true.

By using the Four Properties Theorem, we get

\[x^2 - 3x + 2 = 0\]

is true.

By factoring the left side of the above equation,

\[(x - 1)(x - 2) = 0\]

is true.

Then by using the Zero Product Property Theorem,

\[x - 1 = 0 \text{ or } x - 2 = 0.\]

\(^2\text{You should have learned both the multiplicative inverse property and the multiplicative identity property in K-12.}\)
By using the Four Properties Theorem again, we have narrowed down the possible solutions to just two numbers, namely

\[ x = 1 \text{ or } x = 2. \]

The above two statements are true only under our assumption that the given statement is true for some number \( x \). So, we must check if these results agree with our assumption.

When \( x = 1 \), we run into the problem of division by zero. That is, the given equation is meaningless when \( x = 1 \) and therefore, cannot be true. Therefore 1 is not a solution of the given equation.

When \( x = 2 \),

the left side of the equation \( \frac{2}{2+1} - \frac{2}{2-1} = \frac{2}{3} - 2 = -\frac{4}{3} \),

and the right side of the equation \( \frac{-4}{(2^2-1)} = -\frac{4}{3} \).

Therefore, 2 is a solution of the given equation.

\[ \square \]

### 1.3 Radical Equations

You may have seen the following theorems in K-12.

**Theorem.** Suppose \( a, b \) are real numbers. If \( a = b \), then \( a^2 = b^2 \).

**Proof.** It is given that \( a = b \). Then by the Four Properties Theorem, \( a \cdot a = a \cdot b \). That is, \( a^2 = ab \). Since \( a = b \), we can replace the \( a \) on the right side of the equation by \( b \). Therefore, \( a^2 = b^2 \).

\[ \square \]

**Theorem.** Suppose \( a, b \) are real numbers. If \( a = b \), then \( a^3 = b^3 \).

**Proof.** It is given that \( a = b \). Then by the previous theorem \( a^2 = b^2 \). By the Four Properties Theorem, \( a \cdot a^2 = a \cdot b^2 \). That is \( a^3 = ab^2 \). Since \( a = b \), we can replace the \( a \) on the right side of the equation by \( b \). Therefore, \( a^3 = b^3 \).

\[ \square \]
1.3. RADICAL EQUATIONS

Now it should be easy to see why the following theorem is true.

**Theorem** (Theorem 2). Suppose $a$, $b$ are real numbers and $n$ is a positive integer so that $n \geq 2$. If $a = b$, then $a^n = b^n$.

Solutions to any rational equation can be found by using the method demonstrated in the following theorem.

**Theorem.** The only solution of the equation $\sqrt{3x + 3} + \sqrt{x + 2} = 5$ is 2.

**Proof.** Suppose $\sqrt{3x + 3} + \sqrt{x + 2} = 5$ is true for some real number $x$. Then by the Four Properties Theorem,

$\sqrt{3x + 3} = 5 - \sqrt{x + 2}$

is true. Then by Theorem 2,

$(\sqrt{3x + 3})^2 = (5 - \sqrt{x + 2})^2$

is true. Then by using the Binomial Square Identity on the right side,

$3x + 3 = 5^2 - 2(5)\sqrt{x + 2} + (\sqrt{x + 2})^2$

is true. That is,

$3x + 3 = 25 - 10\sqrt{x + 2} + x + 2$

is true. Then by the Four Properties Theorem and by collecting like terms,

$2x - 24 = -10\sqrt{x + 2}$

is true. We can divide both sides by 2 by using the Four Properties Theorem, just to get a slightly less complicated equation. That is,

$x - 12 = -5\sqrt{x + 2}$

is true. By using Theorem 2,

$(x - 12)^2 = (-5\sqrt{x + 2})^2$
is true. That is,
\[ x^2 - 24x + 144 = 25(x + 2) \]
is true. By using the Four Properties Theorem and collecting like terms,
\[ x^2 - 49x + 94 = 0 \]
is true. By using the Four Properties Theorem,
\[ x^2 - 49x = -94 \]
is true. By completing the square,
\[ x^2 - 49x + \left(\frac{49}{2}\right)^2 = -94 + \left(\frac{49}{2}\right)^2 \]
is true. That is,
\[ \left(x - \frac{49}{2}\right)^2 = \frac{2025}{4} \]
is true. By theorem 1,
\[ x = \frac{49}{2} + \sqrt{\frac{2025}{4}} \text{ or } x = \frac{49}{2} - \sqrt{\frac{2025}{4}}. \]
That is,
\[ x = 47 \text{ or } x = 2. \]
These results are obtained by making the assumption that \( \sqrt{3x + 3} + \sqrt{x + 2} = 5 \) is a true statement. Therefore, we must check and see if our assumption is true.

If \( x = 47 \), then the left side of the equation is \( \sqrt{3(47)} + 3 + \sqrt{47} + 2 = \sqrt{144} + \sqrt{49} = 12 + 7 = 19 \). However, the right side is 5. Therefore, 47 is not a solution of the equation.

If \( x = 2 \), then the left side of the equation is \( \sqrt{3(2)} + 3 + \sqrt{2} + 2 = \sqrt{9} + \sqrt{4} = 3 + 2 = 5 \). Therefore, 2 is a solution of the equation.
1.4 Equations That can be Written as Quadratic Equations

We know that \( x^2 - 2x - 3 = 0 \) is a quadratic equation in \( x \). Similarly, we say \( (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0 \) is a quadratic equation in \( \sqrt{x} \). We say \( (x^2)^2 - 2x^2 - 3 = 0 \) is a quadratic equation in \( x^2 \). We say \( (2x + 1)^2 - 2(2x + 1) - 3 = 0 \) is a quadratic equation in \( 2x + 1 \).

We know how to solve a general quadratic equation by completing the square. Therefore, if we can identify an equation as a quadratic equation, then we know how to solve it. The following example demonstrates the method of finding the solutions of those equations.

**Example.** Find the solutions of the equation \( x - 2\sqrt{x} - 3 = 0 \).

**Solution.** Once again we will assume that \( x - 2\sqrt{x} - 3 = 0 \) is true for some real number \( x \). (A good student should realize that \( x \geq 0 \) to begin with.) Notice that the given equation can be written as \( (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0 \). Therefore, it is a quadratic equation in \( \sqrt{x} \). If we let \( u = \sqrt{x} \), then the given equation is \( u^2 - 2u - 3 = 0 \). Then either by completing the square or by factoring, we can show that the solutions of \( u^2 - 2u - 3 = 0 \) are \( u = 3 \) and \( u = -1 \). That means \( \sqrt{x} = 3 \) or \( \sqrt{x} = -1 \). At this point you should realize that the second possibility, that is, \( \sqrt{x} = -1 \), is not valid since \( \sqrt{x} \geq 0 \) for any non-negative real number \( x \). By using Theorem 2 on the first possibility, \( x = 9 \). Now we check and see if our assumption is true. If \( x = 9 \), then \( 9 - 2\sqrt{9} - 3 = 9 - 2(3) - 3 = 0 \). Therefore, the only solution of \( x - 2\sqrt{x} - 3 = 0 \) is 9.

1.5 Absolute-Value Inequalities

Students may have seen how to solve absolute value inequalities by algebraic methods in K-12. However, the following geometric method is visually superior and will help students understand the definition of a limit of a function when they take calculus.

Solving an inequality in \( x \) means finding all numbers \( x \) that would make the given inequality true. For example, the inequality \( x \geq 2 \) is true for all numbers \( x \) greater than or equal to 2. Therefore, we usually write the solutions of inequalities in interval notation. That is, the solutions of the inequality \( x \geq 2 \) are given by the interval \([2, \infty)\).
Definition. For any real number \( x \), the absolute value of \( x \), denoted by \(|x|\) is:

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

You can remember this definition as follows: To get the absolute value of a number \( x \), if \( x \) is positive or 0, then keep it. If \( x \) is negative, then change the sign.

The method described in the following example demonstrates how to solve an absolute value inequality geometrically.

One Dimensional Distance Formula Theorem. Let \( x \) and \( a \) be two real numbers. Then the distance between \( x \) and \( a \) on the number line is \(|x - a|\).

Exercise. Prove the One Dimensional Distance Formula Theorem.

We will use the One Dimensional Distance Formula Theorem to solve absolute value inequalities.

Example. Solve \(|x - 2| < 3\). (“Solve” means “find the solutions of”.)

Solution. By the One Dimensional Distance Formula Theorem, we can restate the given inequality by translating it into everyday English.

We want “all values of \( x \) so that the distance between \( x \) and 2 is less than 3 units.”

We do not know what (number or numbers) \( x \) will satisfy this claim (yet). But we could plot the point 2 on a number line as follows.

Furthermore, we know that \( x \) should lie within 3 units from 2. So we will identify two points 3 units from 2 as follows.
The three points $-1$, $2$ and $5$ partition the line into the intervals $(-\infty, -1)$, $(-1, 2)$, $(2, 5)$, $(5, \infty)$ and the points $-1$, $2$ and $5$. (All these sets have no points in common.)

Now we check each of these sets for the location of $x$ to see if the given inequality is true.

If $x$ is in $(-\infty, -1)$, then the distance between $x$ and $2$ is more than $3$. See the following picture.

If $x$ is in $(-1, 2)$, then the distance between $x$ and $2$ is less than $3$. See the following picture.

If $x$ is in $(2, 5)$, then the distance between $x$ and $2$ is less than $3$. See the following picture.

If $x$ is in $(5, \infty)$, then the distance between $x$ and $2$ is more than $3$. See the following picture.

If $x = -1$, then the distance between $x$ and $2$ is $3$. See the following picture.
CHAPTER 1. EQUATIONS AND INEQUALITIES

If \( x = 2 \), then the distance between \( x \) and 2 is 0 (less than 3). See the following picture.

If \( x = 5 \), then the distance between \( x \) and 2 is 3. See the following picture.

Now we will highlight the intervals, based on our observations, where the given inequality is true.

Therefore, if \( x \) is in the interval \((-1, 5)\), then the inequality is true. Therefore, the solutions of the inequality lie in the open interval \((-1, 5)\). We loosely say, the solution is the interval \((-1, 5)\).

You may have learned the following theorem on inequalities in K-12.

**Theorem** (Theorem 3). Suppose \( a \), \( b \) and \( c \) are real numbers.

1. If \( a > b \), then \( a + c > b + c \).
2. If \( a > b \), then \( ac > bc \), if \( c > 0 \).  
3. If \( a > b \), then \( ac < bc \), if \( c < 0 \).

You may have learned the following theorem on inequalities in high school.

**Theorem** (Theorem 4). If \( a \) and \( b \) are real numbers, then \( |ab| = |a||b| \).

With the help of Theorems 3 and 4, we can find the solutions of slightly more complicated absolute value inequalities.

Example. Solve $|4x - 8| < 12$.

Solution. The left side of the given inequality is the same as $|4(x - 2)|$, by the distributive law. Then by Theorem 4, this is the same as $|4||x - 2|$. But $|4| = 4$, by definition. Therefore, the given inequality can be written as $4|x - 2| < 12$. By Theorem 3, if $4|x - 2| < 12$ is true, then $|x - 2| < 3$ is true. (Multiply both sides by $\frac{1}{4}$.) Also, by the same theorem, if $|x - 2| < 3$ is true, then $4|x - 2| < 12$ is true. (Multiply both sides by $4$.) Therefore, if we can find the solutions of $|x - 2| < 3$, then they will be the solutions of $4|x - 2| < 12$ and vice versa. At this point, you will use the geometric method that you learned in the previous example to solve $|x - 2| < 3$.

1.6 Quadratic Formula

Since by now students have had plenty of practice with completing the square and are familiar with that method, this should be a good time to prove the Quadratic Formula Theorem for real numbers.

**Quadratic Formula Theorem.** Suppose $a \neq 0$, $b$, and $c$ are fixed real numbers (constants), and $x$ is any real number. Then the solutions of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

provided $b^2 - 4ac \geq 0$.

If $b^2 - 4ac = 0$, then the two solutions are equal.

Convention: It is customary to write the two solutions together as $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Proof. Suppose $ax^2 + bx + c = 0$ is true for some real number $x$. Then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$
is true. Then
\[ x^2 + \frac{b}{a}x = -\frac{c}{a} \]
is true. By completing the square,
\[ x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \]
is true. That is,
\[ \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \]
is true. That means
\[ \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0 \]
is true. Then
\[ \left[\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}\right] \left[\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}\right] = 0 \]
is true, provided \(b^2 - 4ac \geq 0\), by using the Difference of Squares identity. That is,
\[ \left[x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right] \left[x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right] = 0 \]
is true. Now, by the Zero Product Property Theorem,
\[ x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]
Now we check and see if our assumption is true for these numbers.
If \(x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\), then
\[ ax^2 + bx + c = a \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)^2 + b \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) + c \]
\[ = \frac{1}{4a^2} \left[a \left(-b - \sqrt{b^2 - 4ac}\right)^2 + 2ab \left(-b - \sqrt{b^2 - 4ac}\right) + 4a^2c\right] \]
\[ = \frac{1}{4a^2} \left[a \left(b^2 + 2b\sqrt{b^2 - 4ac} + b^2 - 4ac\right) + 2ab \left(-b - \sqrt{b^2 - 4ac}\right) + 4a^2c\right] \]
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\[ = \frac{1}{4a^2} \left[ ab^2 + 2ab\sqrt{b^2 - 4ac} + a^2b - 4a^2c - 2ab^2 - 2ab\sqrt{b^2 - 4ac} + 4a^2c \right] \]
\[ = \frac{1}{4a^2} \left[ 0 \right] \]
\[ = 0 \]

Therefore, \( x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \) is a solution of \( ax^2 + bx + c = 0 \).

The way of checking the other solution is similar: you should be able to verify that
\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

is also a solution of \( ax^2 + bx + c = 0 \).

\[ \square \]

**Note 1** The proof of the previous theorem is valid only if \( b^2 - 4ac \geq 0 \). This is because \( \sqrt{b^2 - 4ac} \) has no real meaning when \( b^2 - 4ac < 0 \). If \( b^2 - 4ac < 0 \), then the equation has no real solutions.

**Note 2** If \( b^2 - 4ac = 0 \), then both solutions are the same. In this case, the solution is \( \frac{-b}{2a} \).

**Note 3** The Quadratic Formula Theorem can be used to factor\(^3\) quadratic polynomials rather quickly.

Suppose \( a \) is a positive whole number and \( b \) and \( c \) are integers. We want to factor \( ax^2 + bx + c \) so that the resulting linear factors are of the form \( rx + s \), where \( r \) is a positive integer and \( s \) is an integer.

Let \( d^2 = b^2 - 4ac \). Calculate \( d^2 \). If \( d^2 \) is negative or if \( d^2 \) is not a perfect (whole number) square, then the given trinomial is not factorable.

If \( d^2 \) is a perfect square, then pick the square-root of \( d^2 \) as \( d \). For example, if \( d^2 = 16 \), then pick \( d = 4 \). Then, compute \( \frac{-b+d}{2a} \) and \( \frac{-b-d}{2a} \). If \( \frac{-b+d}{2a} \) and \( \frac{-b-d}{2a} \) are integers, then the factors of the trinomial are \( \left( x - \frac{-b+d}{2a} \right) \) and \( \left( x - \frac{-b-d}{2a} \right) \).

If \( \frac{-b+d}{2a} \) and \( \frac{-b-d}{2a} \) are rational numbers, then suppose \( \frac{-b+d}{2a} = \frac{k}{\ell} \) and \( \frac{-b-d}{2a} = \frac{m}{n} \) in reduced form, where \( \ell \) and \( n \) are whole numbers. Then \( (\ell x - k) \) and \( (nx - m) \) are the factors of the trinomial.

For example, if you wish to factor \( 6x^2 - 4x + 5 \), then \( b^2 - 4ac = 16 - 4(6)(5) < 0 \). Therefore, \( 6x^2 - 4x + 5 \) cannot be factored. If you wish to factor \( 6x^2 - 4x - 5 \),

\(^3\)Here “factor” means factoring over the integers just like in high school. That is, the coefficients of the factors are integers. For example \( x^2 - 4 \) is factorable over the integers as \((x - 2)(x + 2)\), but \( x^2 - 3 \) is not factorable over the integers.
then \( b^2 - 4ac = 16 + 4(6)(5) = 136 \). But 136 is not a perfect square. Therefore, \( 6x^2 - 4x - 5 \) is not factorable either. If you wish to factor \( 6x^2 - 5x - 6 \), then \( b^2 - 4ac = 25 + 4(6)(6) = 169 \). This is a perfect square, and \( d = 13 \). Then, \( \frac{-b+d}{2a} = \frac{3}{2} \) and \( \frac{-b-d}{2a} = -\frac{2}{3} \) in reduced form. Therefore, the factors are \((2x - 3)\) and \((3x + 2)\).

**Note 4** If we are interested in factoring over real numbers, then \[
\left[ x - \left( -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) \right]
\text{ and }
\left[ x - \left( -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right) \right]
\] are factors of \( ax^2 - bx + c \), if \( b^2 - 4ac > 0 \).

**Note 5** Since \( b^2 - 4ac \) is such an important quantity for a given quadratic polynomial \( ax^2 + bx + c \), we will call it the *discriminant* of \( ax^2 + bx + c \).

**Note 6** If the discriminant of a polynomial \( ax^2 + bx + c \) is negative, then the number \( ax^2 + bx + c \neq 0 \) for any real number \( x \). That means, according to the Trichotomy Law for real numbers, \( ax^2 + bx + c \) is either positive or negative for any real \( x \).

We can do better with the use of following two properties of real numbers.

If \( a \) is a real number, then \( a + 0 = a \). This is known as the *Additive Identity Property* for real numbers.

If \( a \) is a real number, then \( a + (-a) = 0 \). This is known as the *Additive Inverse Property* for real numbers.

By using the above two properties and completing the square, we can write \( ax^2 + bx + c \) in an equivalent form that sheds more light onto the collection of numbers \( ax^2 + bx + c \), for all real numbers \( x \).

\[
ax^2 + bx + c = a \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) \quad \text{by the distributive property.}
\]

\[
= a \left( x^2 + \frac{b}{a}x + \frac{c}{a} + 0 \right) \quad \text{by the additive identity property.}
\]

\[
= a \left( x^2 + \frac{b}{a}x + \frac{c}{a} + \left( \frac{b}{2a} \right)^2 - \left( \frac{b}{2a} \right)^2 \right) \quad \text{by the additive inverse property.}
\]
\[ a \left( x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} \]

by the distributive property.

\[ a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} \]

By the Squares are Nonnegative Theorem, \((x + \frac{b}{2a})^2 \geq 0\). That is, if \(a > 0\), then the smallest value of \(a \left( x + \frac{b}{2a} \right)^2\) is 0. If \(a < 0\), then by Theorem 3, part 3, the largest value of \(a \left( x + \frac{b}{2a} \right)^2\) is 0.

Observe that \(\frac{4ac - b^2}{4a}\) is a constant. Therefore, if \(a > 0\), then the smallest value of \(ax^2 + bx + c\) is \(\frac{4ac - b^2}{4a}\), and if \(a < 0\), then the largest value of \(ax^2 + bx + c\) is \(\frac{4ac - b^2}{4a}\).

Suppose \(a > 0\) and \(b^2 - 4ac < 0\). Then \(\frac{4ac - b^2}{4a} > 0\). That is, the smallest value of \(ax^2 + bx + c\) is positive. Since \(ax^2 + bx + c \neq 0\), this means that \(ax^2 + bx + c > 0\) for all \(x\).

Suppose \(a < 0\) and \(b^2 - 4ac < 0\). Then \(\frac{4ac - b^2}{4a} < 0\). That is, the largest value of \(ax^2 + bx + c\) is negative. Since \(ax^2 + bx + c \neq 0\), this means that \(ax^2 + bx + c < 0\) for all \(x\).

### 1.7 Polynomial and Rational Inequalities

We will assume that one side of the inequality is 0. We will also assume that any polynomial of degree greater than 2 in a polynomial or rational inequality is factorable. In other words, we will look only at factorable inequalities where one side is equal to 0. The method demonstrated in the following example can be used for those types of inequalities.

**Example.** Solve

\[ \frac{(x^2 - x + 1)(x - 2)^2(x + 3)}{x^2 - 3} \leq 0, \]

if possible.

**Solution.** We will again use a geometrical argument to solve this inequality. First, we want to make sure the rational expression on the left side is completely factored over the reals. The polynomial \(x^2 - x + 1\) is not factorable because the discriminant is negative.
We say such a quadratic polynomial irreducible over the reals. However, $x^2 - 3$ can be factored using the Difference of Squares identity into $(x - \sqrt{3})(x + \sqrt{3})$. Therefore, the given inequality can be written as:

$$\frac{(x^2 - x + 1)(x - 2)^2(x + 3)}{(x - \sqrt{3})(x + \sqrt{3})} \leq 0$$

The left side is a product (or quotient) of linear polynomials or irreducible quadratic polynomials. By the Trichotomy Law for real numbers, each polynomial as a number is either positive, negative or zero. The points where each polynomial is equal to zero are called the critical points of the inequality. In this case, $2$, $\sqrt{3}$ and $-\sqrt{3}$ are the critical points. Mark those critical points on a number line as shown below.

We have found the points where each linear polynomial is zero. For example, $x - 2$ is zero when $x = 2$. So, for all other points on the number line, $x - 2$ is either positive or negative. When $x < 2$, by Theorem 3, $x - 2 < 0$ and when $x > 2$, by the same theorem, $x - 2 > 0$. We can include this information below the same number line as follows.

In a similar way, $x + 3 < 0$ when $x < -3$ and $x + 3 > 0$ when $x > -3$. 

$(x - \sqrt{3}) < 0$ when $x < \sqrt{3}$ and $(x - \sqrt{3}) > 0$ when $x > \sqrt{3}$. Also, $(x + \sqrt{3}) < 0$ when $x < -\sqrt{3}$ and $(x + \sqrt{3}) > 0$ when $x > -\sqrt{3}$. 
1.7. POLYNOMIAL AND RATIONAL INEQUALITIES

The discriminant of the (irreducible) quadratic factor \(x^2 - x + 1\) is negative and the leading coefficient is positive. Therefore, by Note 6 following the Quadratic Formula Theorem, \(x^2 - x + 1 > 0\) for all \(x\).
At \( -\sqrt{3} \), the left side of the inequality is undefined.

At \( \sqrt{3} \), the left side of the inequality is undefined.

At 2, the left side of the inequality is 0.

Putting all this information together, we see that the given inequality is true for the intervals shown in the following figure.

Therefore, the solution to the given inequality is \((-3, -\sqrt{3}] \cup [\sqrt{3}, 2]\).
Chapter 2

Graphs of Polynomial and Rational Functions

2.1 Graphs of Quadratic Functions

Given a (real-valued) function \( f \), the collection of all points \((x, f(x))\), for each point \( x \) in the domain of \( f \) is called the graph of \( f \).

We will assume that you know the definition of a real-valued function. We will also assume that you have sketched graphs of basic functions such as linear functions, \( f(x) = x^2 \), \( f(x) = x^3 \), \( f(x) = |x| \), and \( f(x) = \frac{1}{x} \) by using several well-chosen points on the graph by creating tables.

In this section, we will sketch the graphs of quadratic functions by looking at attributes we will call “the end behavior”, “the zeros”, and “the vertex”.

The behavior of the function for very large values of \( x \) or very small values of \( x \) is called the end behavior. We indicate “very large values of \( x \)” by using the symbols \( x \to \infty \) and “very small values of \( x \)” by using the symbols \( x \to -\infty \).

A value of \( x \) at which \( f(x) = 0 \) is called a zero of \( f \).

A point on the graph of a quadratic equation at which the \( f \) has the smallest value or the largest value is called the vertex of the graph of the quadratic function \( f \).

Our options are limited when comes to graphs in Precalculus. The concepts of continuity, increasing, decreasing, and concavity are not available. Therefore, our approach here will be to rely on some known properties of graphs of a few known functions to sketch the
graphs of more sophisticated functions. Our motto will be not to add any extra wiggles or extra turns or introduce gaps or holes to a graph without providing a reason. With that in mind we will start with basic quadratic functions.

**Example.** Sketch the graph of $f(x) = x^2$.

**Solution.** End behavior:
For $x \to \infty$, $f(x) = x^2 \to \infty$. (For large values of $x$, $x^2$ is large.)
For $x \to -\infty$, $f(x) = x^2 \to \infty$. (For $-\text{(large values of } x\text{), } x^2 \text{ is large.}$)

Zeros:
$f(x) = 0$ when $x = 0$. Therefore, 0 is the zero of $f$.

Vertex:
We know that $x^2 \geq 0$, for any real number $x$. Therefore, $f$ has the smallest value when $x = 0$. That is, $(0, f(0))$ is the point on the graph where $f$ is minimum. That is, $(0,0)$ is the vertex.

Now sticking to our declared motto of not adding any extra wiggles or extra turns or introducing gaps or holes to a graph without providing a reason, (and also relying on past experience), we will sketch the graph of $f(x) = x^2$.

By going through the same analysis, that is, finding the end behavior, finding the zero, finding the vertex, and sketching the graph following our established motto, you will see that the graph of $g(x) = ax^2$, where $a$ is a positive constant, is similar to the graph of $f(x) = x^2$. 
2.1. GRAPHS OF QUADRATIC FUNCTIONS

Therefore, we say graphs of \( f_1(x) = x^2 \), \( f_2(x) = \frac{13}{245}x^2 \), \( f_3(x) = 4672x^2 \) have the “same shape”.

Example. Sketch the graph of \( f(x) = -x^2 \).

Solution. End behavior:
For \( x \rightarrow \infty \), \( f(x) = -x^2 \rightarrow -\infty \).
For \( x \rightarrow -\infty \), \( f(x) = x^2 \rightarrow -\infty \).

Zeros:
\( f(x) = 0 \) when \( x = 0 \). Therefore, 0 is the zero of \( f \).

Vertex:
We know that \(-x^2 \leq 0\), for any real number \( x \). Therefore, \( f \) has the largest value when \( x = 0 \). That is, \( (0,f(0)) \) is the point on the graph where \( f \) is maximum. That is, \((0,0)\) is the vertex.

The graph of \( f(x) = -x^2 \) is given below.

The graph of \( g(x) = ax^2 \), where \( a \) is a negative constant, is similar to the graph of \( f(x) = -x^2 \).

Example. Sketch the graph of \( f(x) = 234(x - 57)^2 \).

Solution. End behavior:
For \( x \rightarrow \infty \), that is, for large \( x \), \( f(x) \approx 234x^2 \). (You can think of this in terms of money.)
Suppose $x$ is one billion dollars. (Bill Gates has more than one billion dollars.) Now, if you subtract 57 dollars from a billion dollars, then the remaining number, for all practical purposes, is still roughly a billion dollars. If Bill Gates misplaces 57 dollars, he may not even notice it.) Therefore, the end behavior of $f$ is the same as the end behavior of $g(x) = ax^2$, where $a$ is a positive constant.

Zeros:
$f(x) = 0$ when $x = 57$. Therefore, 57 is the zero of $f$.

Vertex:
We know that $(x - 57)^2 \geq 0$, for any real number $x$. Therefore, $f$ has the smallest value when $x = 57$. That is, $(57, f(57))$ is the point on the graph where $f$ is minimum. That is, $(57, 0)$ is the vertex.

The graph of $f(x) = 234(x - 57)^2$ is given below.

The graph of $g(x) = a(x - h)^2$, where $a$ is a positive constant and $h$ is a constant, is similar to the graph of $f(x) = 234(x - 57)^2$. That is, the graphs of $f_1(x) = \frac{2465}{345}(x - (-48))^2$, $f_2(x) = 973(x - 6893)^2$ have the “same shape” as the graph of $f(x) = 234(x - 57)^2$.

Similar analysis leads to the following graph of the function $f(x) = a(x - h)^2$, where $a$ is a negative constant and $h$ is a constant.
Example. Sketch the graph of $f(x) = 3481(x - 545)^2 + 38$.

Solution. End behavior:
For $x \to \infty$, $f(x) \approx 3481x^2$. (For large $x$, $x - 545 \approx x$ and then for large $x$, $3481x^2 + 38 \approx 3481x^2$.) Therefore, the end behavior of $f(x) = 3481(x - 545)^2 + 38$ is the same as the end behavior of $g(x) = 3481x^2$.

Vertex:
We know that $(x - 545)^2 \geq 0$, for any real number $x$. Therefore, $f$ has the smallest value when $x = 545$. That is, $(545, f(545))$ is the point on the graph where $f$ is minimum. That is, $(545, 38)$ is the vertex.

zeros:
If for some $x$, $f(x) = 0$, then $3481(x - 545)^2 + 38 = 0$. But this is impossible because the left side of this equation is positive for any real number $x$. Therefore, the equation, $3481(x - 545)^2 + 38 = 0$, has no solutions. (This is an instance where the skills developed in solving quadratic equations are helpful.) That is, $f$ has no zeros.

The graph of $f$ is given below.
The graph of \( g(x) = a(x-h)^2 + k \), where \( a \) is a positive constant and \( h \) and \( k \) are constants is similar to the graph of \( f(x) = 3481(x-545)^2 + 38 \).

The graph of \( f(x) = a(x-h)^2 + k \), where \( a \) is a negative constant, and \( h \) and \( k \) are constants, has a graph similar to the following graph.

\( f(x) = a(x-h)^2 + k \) is called the standard form of a quadratic function. \( f(x) = ax^2 + bx + c \) is called the general form of a quadratic function. Based on our experience, we can
quickly sketch the graph of a quadratic function and identify the vertex if it is given in the standard form. In Note 6 to the Quadratic Formula Theorem you have seen how to convert a quadratic polynomial in general form to standard form by completing the square.

Example. Identify the end behavior, find the vertex, find the zeros, and sketch accurately the graph of \( f(x) = 32x^2 - 38x + 41 \).

Solution.

\[
32x^2 - 38x + 41 = 32 \left( x^2 - \frac{38}{32}x + \frac{41}{32} \right) \\
= 32 \left( \left( x^2 - \frac{19}{16}x + \left( \frac{19}{32} \right)^2 \right) + \frac{41}{32} - \left( \frac{19}{32} \right)^2 \right) \\
= 32 \left( x - \frac{19}{32} \right)^2 + 41 - \frac{19^2}{32} \\
= 32 \left( x - \frac{19}{32} \right)^2 + \frac{951}{32}
\]

The end behavior:
For \( x \to \infty \), \( f(x) \approx 32x^2 \). Therefore,
For \( x \to \infty \), \( f(x) \to \infty \).
For \( x \to -\infty \), \( f(x) \to \infty \).

The vertex:
The function has the smallest value when \( x = \frac{19}{32} \). Therefore, the vertex is \( \left( \frac{19}{32}, f\left( \frac{19}{32} \right) \right) \).
That is, the vertex is \( \left( \frac{19}{32}, \frac{951}{32} \right) \).

If we assume that \( f(x) = 0 \) for some value of \( x \), then we get \( 32 \left( x - \frac{19}{32} \right)^2 + \frac{951}{32} = 0 \). However, the left side of this equation is positive for all real numbers \( x \). Therefore, this equation has no solutions. That is, this function has no zeros.

The graph of the function is:
Exercise. A toy manufacturer thinks that he can minimize the cost by making less toys than he makes now in his factory. The factory produces 1000 toys per day. A consultant hired by the manufacturer calculated the cost $C$ in dollars to produce $x$ toys per day as $C(x) = 15x^2 - 29430x + 14435527$. Without looking at the rest of the consultant’s report, can you figure out if the thinking of the manufacturer is correct? If so, how many toys should he make per day to minimize the cost?

### 2.2 Graphs of Polynomial Functions

As mentioned before, we cannot accurately sketch the graph of a polynomial function without the tools of calculus. However, we can do a reasonably good job, provided that a polynomial function can be factored into linear factors or irreducible quadratic factors. For example, we can get a reasonably accurate graph of

$$f(x) = 241(x - 356)^2(32x - 43)^3(x + 41)(x^2 - x + 41)$$

with the help of known properties of very basic polynomial functions.
With that in mind let us summarize what we have so far.

Exercise. Find the end behavior, the zero, and sketch the graph of \( f(x) = 24(x - 31)^4 \).

It is easy to see that the graph of \( f(x) = a(x - h)^n \), where \( n \) is even and \( a > 0 \), has the same end behavior, the same zero, and the same behavior near zero as the graph of \( f(x) = a(x - h)^2 \), where \( a > 0 \). Also, \( f(x) = a(x - h)^n \), where \( n \) is even and \( a < 0 \), has the same end behavior, the same zero, and the same behavior near zero as the graph of \( f(x) = a(x - h)^2 \), where \( a < 0 \). We summarize our findings in the following figure.
Now we will examine the graphs of cubic polynomials.

**Example.** Sketch the graph of \( f(x) = 217(x - 53)^3 \).

**Solution.** End behavior:
For \( x \to \infty \), \( f(x) \approx 217x^3 \to \infty \).
For \( x \to -\infty \), \( f(x) \approx 217x^3 \to -\infty \).

Zeros:
\( f(x) = 0 \) when \( x = 53 \). Therefore, 53 is the zero of \( f \).

Now sticking to our motto of not adding any extra wiggles or extra turns or introducing gaps or holes to a graph without providing a reason, and definitely relying on past experience, we will sketch the graph of \( f(x) = 217(x - 53)^3 \).

It is easy to see that the graphs of \( f(x) = a(x - h)^n \), where \( n > 3 \) is odd and \( a > 0 \), has the same end behavior, the same zero, and the same behavior near zero as the graph of \( f(x) = 217(x - 53)^2 \).

**Example.** Sketch the graph of \( f(x) = -217(x - 53)^3 \).

**Solution.** End behavior:
For \( x \to \infty \), \( f(x) \approx -217x^3 \to -\infty \).
For \( x \to -\infty \), \( f(x) \approx -217x^3 \to \infty \).

Zeros:
\( f(x) = 0 \) when \( x = 53 \). Therefore, 53 is the zero of \( f \).
Now sticking to our motto of not adding any extra wiggles or extra turns or introducing gaps or holes to a graph without providing a reason, and definitely relying on past experience, we will sketch the graph of \( f(x) = -217(x - 53)^3 \).

It is easy to see that the graphs of \( f(x) = a(x - h)^n \), where \( n > 3 \) is odd and \( a < 0 \) has the same end behavior, the same zero, and the same behavior near zero as the graph of \( f(x) = -217(x - 53)^2 \).

We summarize our findings in the following figure.

(a) \( f(x) = a(x - h)^n \), where \( a > 0 \) and \( n \geq 3 \) is odd

(b) \( f(x) = a(x - h)^n \), where \( a < 0 \) and \( n \geq 3 \) is odd

Figure 2
Surprisingly, the information in the graphs of figures 1 and 2 is sufficient to sketch the graphs of polynomial functions that can be factored into linear factors or irreducible quadratic factors. The method demonstrated in the following example can be used to sketch the graphs of such polynomial functions.

**Exercise.** Identify the end behavior, the zeros, the behavior near each zero, and sketch the graph of
\[
f(x) = 241(x - 356)^2(32x + 43)^3(x + 41)(x^2 - x + 41)
\]
accurately.

**Solution.** On the side, write the standard form of the polynomial \(x^2 - x + 41\). This turns out to be \((x - \frac{1}{2})^2 + \frac{163}{4}\). Now we know that \(x^2 - x + 41 > 0\) for all \(x\). We also know that \(x^2 - x + 41 \approx x^2\) for large values of \(x\). With this additional information, we can begin analyzing the given polynomial function.

**The end behavior:**
For \(x \to \infty\), \(f(x) \approx 241(x^2)(32^3x^3)(x^2) = 241(32)^3x^8\). Therefore, \(f\) has the same end behavior as \(g(x) = a(x - h)^n\), where \(a > 0\) and \(n\) is even. (Here, \(h = 0\).)

\[
\begin{array}{c|c}
 & y \\
\hline
x & \\
\end{array}
\]

**zeros:**
\(f(x) = 0\), when \(x = 356\), \(x = -\frac{43}{32}\), and \(x = -41\). Therefore, the zeros are 356, \(-\frac{43}{32}\), and -41.

Now we check the behavior near each zero.

Near \(x = 356\):
The factor \(32x - 43 \approx 32(356) + 43 > 0\). Then \((32x + 43)^3 \approx (32(356) - 43)^3 > 0\).

The factor \(x + 41 \approx 356 + 41 > 0\). The factor \(x^2 - x + 41 > 0\), for any \(x\). Let \(a = (32(356) + 43)^3(356 + 41)(356^2 - 356 + 41)\). Then \(a > 0\) and \(f(x) \approx a(x - 356)^2\) near the zero \(x = 356\). Therefore, the graph of \(f\) looks like the graph of \(g_1(x) = a(x - 356)^2\) near \(x = 356\).

Near \(x = -\frac{43}{32}\):
The factor \(x - 356 \approx (-\frac{43}{32} - 356) < 0\). But, \((x - 356)^2 \approx (-\frac{43}{32} - 356)^2 > 0\). The factor
The factor \( x^2 - x + 41 \approx (\frac{-43}{32})^2 - (\frac{-43}{32}) + 41 > 0 \). Let \( b = \frac{32(241)(-\frac{43}{32} + 356)^2(-\frac{43}{32} + 41)((-\frac{43}{32})^2 - (\frac{-43}{32}) + 41)}{2} \). Then \( b > 0 \) and \( f(x) \approx b(x + \frac{43}{32})^3 \). Therefore, the graph of \( f \) looks like the graph of \( g_2(x) = b(x + \frac{43}{32})^3 \) near \( x = -\frac{43}{32} \).

Near \( x = -41 \): The factor \( x - 356 \approx (-41 - 356) < 0 \). But, \( (x - 356)^2 \approx (-41 - 356)^2 > 0 \). The factor \( 32x + 43 \approx 32(-41) + 43 < 0 \) Then \( (32x + 43)^3 \approx (32(-41) + 43)^3 < 0 \). The factor \( x^2 - x + 41 \approx (-41)^2 - (-41) + 41 > 0 \). Let \( c = \frac{241(-41 - 356)^2(32(-41) + 43)^3((-41)^2 - (-41) + 41)}{3} \). Then \( c < 0 \) and \( f(x) \approx c(x + 41) \). Therefore, the graph of \( f \) looks like the graph of the line \( g_3(x) = c(x + 41) \), with a negative slope, near \( x = -41 \).

Now sticking to our motto of not introducing any extra wiggles, gaps or holes without providing reasons, we can sketch the graph of the given function.

### 2.3 Graphs of Rational Functions

In sketching graphs of rational functions, we will use the same method of sketching graphs of functions based on the properties of few simple functions, just as we did with the graphs of polynomial functions. However, there are a few additional properties that we
have to worry about when we are dealing with rational functions, compared to polynomial functions. Let us start with the graph of a simple rational function.

**Example.** Sketch the graph of \( f(x) = \frac{1}{x} \).

**Solution.** *End Behavior:*

For \( x \to \infty \), that is, for large \( x \), \( f(x) = \frac{1}{x} \) is very small but remains positive. We indicate this behavior by using a new notation: \( f(x) \to 0^+ \).

For \( x \to -\infty \), that is, for \(-(large)\) \( x \), \( f(x) = \frac{1}{x} \) is very small but remains negative. We indicate this behavior by using a new notation: \( f(x) \to 0^- \).

**Zeros:**

There is no number \( x \) for which \( f(x) = 0 \). That is, this function has no zeros.

However, \( f \) is undefined at 0. We will check the behavior near 0. We want to see the behavior to the left of 0 and the behavior to the right of 0. We introduce new notations to indicate this behavior.

*The notation* \( x \to 0^- \) *indicates that “* \( x \) *is to the left of 0 but very close to 0”.*

*The notation* \( x \to 0^+ \) *indicates that “* \( x \) *is to the right of 0 but very close to 0”.*

**Near 0:**

For \( x \to 0^- \), \( f(x) < 0 \) and \( |f(x)| \) is very large. We indicate this behavior by the notation \( f(x) \to -\infty \).

For \( x \to 0^+ \), \( f(x) > 0 \) and \( |f(x)| \) is very large. We indicate this behavior by the notation \( f(x) \to \infty \).

With the information we gathered on the graph of \( f \), and with prior experience from high school, now we can sketch the graph by remaining true to our motto.
Exercise.

1. Show that the graph of \( g(x) = \frac{341}{x} \) is similar to the graph of \( f(x) = \frac{1}{x} \) by checking the end behavior, behavior near zeros, and behavior near points where \( g \) is undefined.

2. In general, show that the graph of \( h(x) = \frac{a}{x} \), where \( a \) is a positive constant, is similar to the graph of \( f(x) = \frac{1}{x} \) by checking the end behavior, behavior near zeros and behavior near points where \( g \) is undefined.

The behavior of the graph of \( f(x) = \frac{1}{x} \) near 0 is worth identifying as a special property of \( f \). We say the vertical line \( x = 0 \) is a vertical asymptote of \( f \).

In general, given a function \( f \), if we see at least one of the following behaviors:
for \( x \to a^- \), \( f(x) \to \pm \infty \) or or \( x \to a^+ \), \( f(x) \to \pm \infty \),
then we say that the vertical line \( x = a \) is a vertical asymptote of \( f \).

Example. Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or where the function is undefined, and sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[
f(x) = \frac{341}{x - 23}.
\]
Solution. End Behavior:
For \( x \to \infty \), \( f(x) \approx \frac{341}{x} \). Therefore, the end behavior of \( f \) is the same as the end behavior of \( g_1(x) = \frac{341}{x} \).

Zeros:
\( f \) has no zeros.

Undefined:
\( f \) is undefined at \( x = 23 \).

Behavior near \( x = 23 \):
For \( x \to 23^- \), \( f(x) < 0 \) and \( f(x) \to -\infty \).
For \( x \to 23^+ \), \( f(x) > 0 \) and \( f(x) \to \infty \).

Therefore, the vertical line \( x = 23 \) is a vertical asymptote of \( f \).

A sketch of the graph of \( f \) is given below. The dashed line is not part of the graph of \( f \); it is the vertical asymptote \( x = 23 \). It is customary to include the graph of the vertical asymptote with the graph of the function for clarity.

Exercise. Show that the graph of \( g(x) = \frac{a}{x-b} \), where \( a \) is a positive constant and \( b \) is a constant, is similar to the graph of \( f(x) = \frac{341}{x-23} \) by checking the end behavior, behavior near zeros, and behavior near points where \( g \) is undefined. Show that the vertical asymptote of \( g \) is the line \( x = b \).
Example. Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or where the function is undefined, and sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[ f(x) = \frac{-341}{x - 23}. \]

Solution. End Behavior:
For \( x \to \infty \), \( f(x) \approx \frac{-341}{x} \). For \( x \to \infty \), \( \frac{-341}{x} \) is negative and \( \frac{-341}{x} \to 0^- \).

For \( x \to -\infty \), that is, for -(large) \( x \), \( f(x) \approx \frac{-341}{x} \). For \( x \to -\infty \), \( \frac{-341}{x} \) is positive and \( \frac{-341}{x} \to 0^+ \).

Zeros:
\( f \) has no zeros.

Undefined:
\( f \) is undefined at \( x = 23 \).

Behavior near \( x = 23 \):
For \( x \to 23^- \), \( f(x) > 0 \) and \( f(x) \to \infty \).

For \( x \to 23^+ \), \( f(x) < 0 \) and \( f(x) \to -\infty \).

The vertical line \( x = 23 \) is a vertical asymptote of \( f \).

A sketch of the graph of \( f \) is given below, including the vertical asymptote.

Exercise. Show that the graph of \( g(x) = \frac{a}{x-b} \), where \( a \) is a negative constant and \( b \) is a constant, is similar to the graph of \( f(x) = \frac{-341}{x-23} \) by checking the end behavior, behavior
near zeros and behavior near points where \( g \) is undefined. Show that the vertical asymptote of \( g \) is the line \( x = b \).

**Example.** Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or the function is undefined, and sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[
f(x) = \frac{341}{(x - 23)^2}.
\]

**Solution.** *End Behavior:*
For \( x \to \infty \), \( f(x) \approx \frac{341}{x^2} \). For \( x \to \infty \), \( \frac{341}{x^2} \) is positive and \( \frac{341}{x^2} \to 0^+ \).

For \( x \to -\infty \), that is, for -(large) \( x \), \( f(x) \approx \frac{341}{x^2} \). For \( x \to -\infty \), \( \frac{341}{x^2} \) is positive and \( \frac{341}{x^2} \to 0^+ \).

**Zeros:**
f has no zeros.

**Undefined:**
f is undefined at \( x = 23 \).

**Behavior near \( x = 23 \):**
For \( x \to 23^- \), \( f(x) > 0 \) and \( f(x) \to \infty \).

For \( x \to 23^+ \), \( f(x) > 0 \) and \( f(x) \to \infty \).

The vertical line \( x = 23 \) is a vertical asymptote of \( f \).

A sketch of the graph of \( f \) is given below, including the vertical asymptote.
Exercise. Show that the graph of \( g(x) = \frac{a}{(x-b)^n} \), where \( a \) is a positive constant, \( b \) is a constant and \( n \) is even, has the same end behavior and the same behavior near the point where \( g \) is undefined as the graph of \( f(x) = \frac{341}{(x-23)^2} \) by checking the end behavior, behavior near zeros and behavior near points where \( g \) is undefined. Show that the vertical asymptote of \( g \) is the line \( x = b \).

Example. Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or where the function is undefined, and sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[ f(x) = \frac{-341}{(x-23)^2} \]

Solution. End Behavior:
For \( x \to \infty \), \( f(x) \approx \frac{-341}{x^2} \). For \( x \to \infty \), \( \frac{-341}{x^2} \) is negative and \( \frac{-341}{x^2} \to 0^- \).

For \( x \to -\infty \), that is, for -(large) \( x \), \( f(x) \approx \frac{-341}{x^2} \). For \( x \to -\infty \), \( \frac{-341}{x^2} \) is negative and \( \frac{-341}{x^2} \to 0^- \).

Zeros:
\( f \) has no zeros.

Undefined:
\( f \) is undefined at \( x = 23 \).

Behavior near \( x = 23 \):
For \( x \to 23^- \), \( f(x) > 0 \) and \( f(x) \to \infty \).

For \( x \to 23^+ \), \( f(x) > 0 \) and \( f(x) \to \infty \).

The vertical line \( x = 23 \) is a vertical asymptote of \( f \).

A sketch of the graph of \( f \) is given below, including the vertical asymptote.
Exercise. Show that the graph of \( g(x) = \frac{a}{(x-b)^n} \), where \( a \) is a negative constant, \( b \) is a constant and \( n \) is even, has the same end behavior, same behavior near the point where \( g \) is undefined as the graph of \( f(x) = \frac{-341}{(x-23)^3} \) by checking the end behavior, behavior near zeros, and behavior near points where \( g \) is undefined. Show that the vertical asymptote of \( g \) is the line \( x = b \).

Example. Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or where the function is undefined, and sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[
f(x) = \frac{341}{(x-23)^3}.
\]

Solution. End Behavior:
For \( x \to \infty \), \( f(x) \approx \frac{341}{x^3} \). For \( x \to \infty \), \( \frac{341}{x^3} \) is positive and \( \frac{341}{x^3} \to 0^+ \).

For \( x \to -\infty \), that is, for -(large) \( x \), \( f(x) \approx \frac{341}{x^3} \). For \( x \to -\infty \), \( \frac{341}{x^3} \) is negative and \( \frac{341}{x^3} \to 0^- \).

Zeros:
\( f \) has no zeros.

Undefined:
\( f \) is undefined at \( x = 23 \).

Behavior near \( x = 23 \):
For \( x \to 23^- \), \( f(x) < 0 \) and \( f(x) \to -\infty \).

For \( x \to 23^+ \), \( f(x) > 0 \) and \( f(x) \to \infty \).

Therefore, the vertical line \( x = 23 \) is a vertical asymptote of \( f \).

A sketch of the graph of \( f \) is given below. The dashed line is not part of the graph of \( f \); it is the vertical asymptote \( x = 23 \). It is customary to include the graph of the vertical asymptote with the graph of the function for clarity.
Exercise. Show that the graph of \( g(x) = \frac{a}{(x-b)^n} \), where \( a \) is a positive constant, \( b \) is a constant, and \( n \) is odd, has the same end behavior and same behavior near the point where \( g \) is undefined as the graph of \( f(x) = \frac{341}{(x-23)} \) by checking the end behavior, behavior near zeros and behavior near points where \( g \) is undefined. Show that the vertical asymptote of \( g \) is the line \( x = b \).

Example. Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or where the function is undefined, and sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[
f(x) = \frac{-341}{(x-23)^3}.
\]

Solution. End Behavior:
For \( x \to \infty \), \( f(x) \approx \frac{-341}{x^3} \). For \( x \to \infty \), \( \frac{-341}{x^3} \) is positive and \( \frac{-341}{x^3} \to 0^+ \).

For \( x \to -\infty \), that is, for -(large) \( x \), \( f(x) \approx \frac{-341}{x^3} \). For \( x \to -\infty \), \( \frac{-341}{x^3} \) is negative and \( \frac{-341}{x^3} \to 0^- \).

Zeros:
\( f \) has no zeros.

Undefined:
\( f \) is undefined at \( x = 23 \).

Behavior near \( x = 23 \):
For \( x \to 23^- \), \( f(x) < 0 \) and \( f(x) \to -\infty \).
For $x \to 23^+$, $f(x) > 0$ and $f(x) \to \infty$.

Therefore, the vertical line $x = 23$ is a vertical asymptote of $f$.

A sketch of the graph of $f$ is given below. The dashed line is not part of the graph of $f$; it is the vertical asymptote $x = 23$. It is customary to include the graph of the vertical asymptote with the graph of the function for clarity.

Exercise. Show that the graph of $g(x) = \frac{a}{(x-b)^n}$, where $a$ is a negative constant, $b$ is a constant and $n$ is odd, has the same end behavior and the same behavior near the point where $g$ is undefined as the graph of $f(x) = \frac{-341}{(x-23)}$ by checking the end behavior, behavior near zeros, and behavior near points where $g$ is undefined. Show that the vertical asymptote of $g$ is the line $x = b$.

We can summarize our findings of basic rational functions in the following four graphs.
2.3. **GRAPHS OF RATIONAL FUNCTIONS**

(a) \( f(x) = \frac{a}{(x-b)^n} \), where \( a > 0 \) and \( n \) is odd

(b) \( f(x) = \frac{-a}{(x-b)^n} \), where \( a < 0 \) and \( n \) is odd

**Figure 3**

(a) \( f(x) = \frac{a}{(x-b)^n} \), where \( a > 0 \) and \( n \) is even

(b) \( f(x) = \frac{-a}{(x-b)^n} \), where \( a < 0 \) and \( n \) is even

**Figure 4**

Let us look at even more general rational functions now.

Let \( f(x) = \frac{g(x)}{h(x)} \) be a rational function. The polynomial \( g(x) \) is called the “numerator” of \( f \) and the polynomial \( h(x) \) is called the “denominator” of \( f \).

We will separate the rational functions into two categories.

**Category 1:** The degree of the numerator is less than the degree of the denominator.
Category 2: The degree of the numerator is greater than or equal to the degree of the denominator.

Let us look at Category 1 first. Let us assume that the given rational function can be factored into linear and irreducible quadratic factors.

With the information in Figures 1, 2, 3, and 4, we can sketch the graphs of such Category 1 functions.

**Example.** Find the end behavior, find the zeros, and find the points where the function is undefined. Check the behavior of the function at the points where the function has a zero or where the function is undefined and, sketch the graph. Include any vertical asymptote of the function on the same set of coordinates.

\[ f(x) = \frac{31(x - 45)(x + 34)^2}{(x - 23)^3(x^2 + 1)}. \]

**Solution.** End Behavior:

For \( x \to \infty \), \( f(x) \approx \frac{31x^3}{x^2} = \frac{31}{x} \). Therefore, the end behavior of \( f(x) \) is the same as the end behavior of \( f(x) = \frac{a}{(x-b)^n} \), where \( a > 0, b = 0 \).

Zeros:
The zeros of \( f \) are 45 and -34.

Behavior near \( x = 45 \):
Near \( x = 45 \), \( f(x) \approx k_1(x - 45) \), where \( k_1 \) is a positive constant. Therefore, the behavior of \( f \) near \( x = 45 \) is like the graph of a line with a positive slope.

(45,0) → \( x \)
Behavior near $x = -34$:
Near $x = -34$, $f(x) \approx k_2(x + 34)^2$, where $k_2$ is a positive constant. Therefore, the behavior of the graph of $f$ near $x = -34$ is like the graph of $g_1(x) = k_2(x + 34)^2$ near $x = 34$.

Undefined:
$f$ is undefined at $x = 23$.

Behavior near $x = 23$:
Near $x = 23$, $f(x) \approx \frac{k_3}{(x - 23)^3}$, where $k_3$ is a negative constant. Therefore, the behavior of the graph $f$ near $x = 23$ is like the behavior of $g(x) = \frac{k_3}{(x - 23)^3}$, $k_3 < 0$.

The vertical line $x = 23$ is a vertical asymptote of $f$.

We can sketch of the graph of $f$ with the collected information and sticking to our motto. A sketch of the graph including the vertical asymptote is given below.
Let us look at Category 2 rational functions now. That is, \( f(x) = \frac{g(x)}{h(x)} \), where the degree of \( g(x) \) is greater than or equal to the degree of \( h(x) \).

**Euclidean Algorithm Theorem.** Given polynomials \( g(x) \) and \( h(x) \), there are unique polynomials \( q(x) \) and \( r(x) \) so that

\[
g(x) = q(x)h(x) + r(x), \text{ where, the degree of } r(x) < \text{ the degree of } h(x).
\]

The following theorem is the same Euclidean Algorithm Theorem when \( h(x) \neq 0 \).

**Euclidean Algorithm Theorem – version 2.** Given polynomials \( g(x) \) and \( h(x) \), there are unique polynomials \( q(x) \) and \( r(x) \) so that

\[
\frac{g(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}, \text{ where the degree of } r(x) < \text{ the degree of } h(x).
\]

We will omit the proof of this theorem. You may have learned the Long Division Algorithm for polynomials in high school. Given \( g(x) \) and \( h(x) \), you can use the Long Division Algorithm to find \( q(x) \) and \( r(x) \). The following example demonstrates the process.

**Example.** If \( g(x) = x^3 - 2x^2 + 4x + 5 \) and \( h(x) = x^2 - 3 \), then find \( q(x) \) and \( r(x) \) that satisfy the Euclidean Algorithm Theorem – version 2.

\[
\begin{array}{c|cccc}
 & x^3 & - x^2 & + 4x & + 5 \\
\hline
 x^2 - 3 & x & - 3x & \\
 & x^3 & - 3x & \\
\hline
 & -x^2 & + 7x & + 5 \\
 & -x^2 & - 3 & \\
\hline
 & 7x & + 2 & \\
\end{array}
\]

Therefore,

\[
\frac{x^3 - x^2 + 4x + 5}{x^2 - 3} = x - 1 + \frac{7x + 2}{x^2 - 3}.
\]
We will first look at the special case of degree of \( p(x) = \text{degree of } q(x) \).

**Example.** Find the end behavior of \( f(x) = \frac{3x+4}{x-1} \).

**Solution.** By using the Long Division Algorithm, we can write \( f(x) \) as:

\[
f(x) = 3 + \frac{7}{x-1}.
\]

Now for \( x \to \infty \), \( \frac{7}{x-1} \to 0^+ \) and for \( x \to -\infty \), \( \frac{7}{x-1} \to 0^- \). In either case, \( f(x) \approx 3 \).

Given a function \( f(x) \), if there is a function \( q(x) \) so that \( f(x) - q(x) \approx 0 \) for \( x \to \pm \infty \), then we say \( y = q(x) \) is a non-vertical asymptote of \( f \).

In the given example, \( y = 3 \) is a non-vertical asymptote of \( f \). Since we know that the graph of \( y = 3 \) is a horizontal line, we usually say \( y = 3 \) is a horizontal asymptote of \( f(x) = \frac{3x+4}{x-1} \).

**Theorem.** Suppose the degree of \( r(x) \) is less than the degree of \( h(x) \). Then for \( x \to \pm \infty \), \( \frac{r(x)}{h(x)} \approx 0 \).

**Exercise.** Prove the previous theorem.

**Theorem.** Let \( f(x) = \frac{g(x)}{h(x)} \). Then the non-vertical asymptote of \( f \) is \( y = q(x) \), where

\[
\frac{g(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}, \text{ and the degree of } r(x) < \text{ the degree of } h(x).
\]

**Proof.** The proof of this theorem follows directly from the Euclidean algorithm and the previous theorem.

The word *corollary* stands for a “little” theorem. Usually the word “corollary” is a little theorem associated with an existing theorem. We use the word corollary for a theorem if it “falls out of” an existing theorem.

The following is a corollary to the previous theorem.
Corollary.

1. If \( f(x) \) is a rational function of Category 1 (that is, the degree of the denominator > the degree of the numerator), then the non-vertical asymptote is a horizontal asymptote, and it is \( y = 0 \).

2. If \( f(x) \) is a rational function of Category 2 (that is, the degree of the denominator \( \leq \) the degree of the numerator), then the non-vertical asymptote is \( y = q(x) \).

Exercise. Prove the previous corollary.

Suppose the function \( f \) is not defined at \( x = a \). Suppose also that there is a real number \( L \) so that for \( x \to a^- \), \( f(x) \approx L \) and for \( x \to a^+ \), \( f(x) \approx L \). Then we say that \( f \) has a hole at \( x = a \).

The following example demonstrates the method of sketching graphs of rational functions of Category 2.

Example. For the given function, find the end behavior, find the zeros, find the asymptotes, find the holes, find the behavior near each zero, find the behavior near each asymptote
CHAPTER 2. GRAPHS OF POLYNOMIAL AND RATIONAL FUNCTIONS

, and sketch the graph accurately.

\[ f(x) = \frac{2x^3 + 3x^2 - 2x}{x^2 - 4} \]

Solution. Since the degree of the numerator is less than the degree of the denominator, (Category 2), we can use the Long Division Algorithm to write this function as

\[ f(x) = 2x + 3 + \frac{6x - 12}{x^2 - 4} \]

Therefore, \( y = 2x + 3 \) is a non-vertical asymptote. This type of a linear asymptote is also known as a slant asymptote.

End behavior: For \( x \to \infty \), \( \frac{6x - 12}{x^2 - 4} \approx \frac{6x}{x^2} = \frac{6}{x} \). Therefore,

For \( x \to \infty \), \( f(x) \approx 2x + 3 + \text{“a very small positive number”}. \)

For \( x \to -\infty \), \( \frac{6x - 12}{x^2 - 4} \approx \frac{6x}{x^2} = \frac{6}{x} \). Therefore,

For \( x \to -\infty \), \( f(x) \approx 2x + 3 + \text{“a very small negative number”}. \)

Now we will factor \( f \) completely.

\[ f(x) = \frac{x(2x - 1)(x + 2)}{(x - 2)(x + 2)}. \]

The factor \((x + 2)\) is common to both numerator and the denominator of \( f(x) \).\(^\text{10}\) Clearly, \( f \) is not defined at \( x = -2 \). Also, for all real numbers \( x \neq -2 \), \( f(x) \) and

\[ g(x) = \frac{x(2x - 1)}{(x - 2)} \]

are identical. This is because, \( \frac{x + 2}{x + 2} = 1 \), for \( x \neq -2 \).

Near \(-2\) (That is, for \( x \to -2^- \) and for \( x \to -2^+ \)):

\[ f(x) \approx \frac{(-2)(2(-2)+1)}{(-2-2)} = -\frac{3}{2}. \]

Therefore, \( f \) has a hole at \( x = -2 \).

Now we will concentrate on \( g \) as \( f(x) = g(x) \) for all \( x \neq -2 \).

Zeros of \( g \) (these are also the zeros of \( f \)):

The zeros of \( g \) are 0 and \( \frac{1}{2} \).

\(^{10}\) is so called an indeterminate form. You may learn about indeterminate forms in a calculus class. If we agree that a division by zero is invalid, no matter what, then we do not have to be concerned about indeterminate forms here.
Near 0:
Near 0, $g(x) \approx \frac{1}{2}x$. Therefore, near 0, the graph of $g$ (and hence the graph of $f$) looks like a line with a positive slope.

Near $\frac{1}{2}$:
Near $\frac{1}{2}$, $g(x) \approx -\frac{1}{3}(2x + 1)$. Therefore, near $\frac{1}{2}$, the graph of $g$ (and hence the graph of $f$) looks like a line with a negative slope.

g is undefined at $x = 2$:
Near $x = 2$, $g(x) \approx \frac{6}{x-2}$. Therefore, the graph of $g$ looks like the graph of $g_1(x) = \frac{6}{x-2}$ near $x = 2$. The vertical asymptote of $g$ is $x = 2$.

By using all the collected information, we can sketch the graph of $f$ now.
Chapter 3

Sequences, Series, Mathematical Induction

3.1 Sequences and Series

We will introduce new notation to represent real numbers, rational numbers, integers and natural numbers. (The natural numbers are the positive integers. The natural numbers are also known as counting numbers.) We will use $\mathbb{R}$ to represent all real numbers; we will use $\mathbb{Q}$ to represent all rational numbers; we will use $\mathbb{Z}$ to represent all integers; and we will use $\mathbb{N}$ to represent all natural numbers. If we want to talk about all except a few real numbers, say “all real numbers except 1 and 2”, then we will use the notation $\mathbb{R} \setminus \{1, 2\}$.

You have seen polynomial functions and rational functions so far. The domain of a polynomial function is $\mathbb{R}$. The domain of a rational function is $\mathbb{R} \setminus \{\text{points where the denominator is zero}\}$.

Now we will look at a special class of real-valued functions whose domain is $\mathbb{N}$. The natural numbers differ from real numbers in many ways. In particular, there is a first natural number, namely, 1. However, there is no first real number. For every natural number, there is a next natural number. For example, the natural number next to 1 is 2. Because of this special nature of natural numbers, there is a first element for a function $f$ with domain $\mathbb{N}$; namely, $f(1)$. The next element to $f(1)$ is $f(2)$. Just like we can list the natural numbers as $\{1, 2, 3, \ldots, k, k+1, \ldots\}$, we can list the elements of $f$ as $\{f(1), f(2), f(3), \ldots, f(k), f(k+1), \ldots\}$, where $k$ is an arbitrary natural number.

Because of this property, we usually call a real-valued function with domain $\mathbb{N}$ a sequence.
The first element of the sequence is \( f(1) \), the second element of the sequence is \( f(2) \), etc.

and the \( k \)th element of the sequence is \( f(k) \), for some arbitrary natural number \( k \).

Another convention is to drop the functional notation when dealing with sequences and just indicate the position of an element using a subscript. Therefore, \( f(1) \) is usually written as \( f_1 \), and \( f(2) \) is usually written as \( f_2 \), etc. But, then you should realize that this new notation conflicts with our previous use of \( f_1, f_2 \) to represent different functions. For this reason, we will replace the letter \( f \) with the letter \( a \) when we are dealing with sequences.

A given sequence can be represented as \( \{a_1, a_2, a_3, \ldots, a_n, \ldots, \} \), where \( a_1 \) is the first element of the sequence, \( a_2 \) is the second element of the sequence, \( a_3 \) is the third element of the sequence, and \( a_n \) is the \( n \)th element of the sequence, for an arbitrary natural number \( n \).

Since sequences are functions, everything we know about functions can be used.

**Example.** Consider \( f(x) = x^2 - 2x - 3 \), one of the familiar quadratic functions. Since we know what the function is, we can find the function value at any given value of \( x \). For example, \( f(\frac{1}{2}) = (\frac{1}{2})^2 - 2(\frac{1}{2}) - 3 \), by the definition of the functional notation that you may have learned in high school.

Now suppose \( a_n = n^2 - 2n - 3 \) is the \( n \)th term of a sequence. Then we can find \( a_5 \) as follows. \( a_5 = 5^2 - 2(5) - 3 \).

As matter of fact, if \( a_n \) is given, then we can find any element of the sequence, just like given \( f(x) \), we can find any function value of \( f \).

For this reason, \( a_n \) is called the **general term** of a sequence. There is no special reason why we should use \( n \) as a subscript of the general term. Any lower case letter other than \( a \) is fine to use as the subscript of the general term of a sequence. But, most of the time, unless there is a very specific reason not to use \( n \), we will use \( a_n \) to represent the general term of a sequence.

**Example.** Consider \( f(x) = \frac{1}{3} \), one of the constant functions. For example, if we want to find what \( f(5) \) is, then it should be clear that \( f(5) = \frac{1}{3} \).

Now suppose \( a_n = \frac{1}{3} \) is the \( n \)th term of a sequence. Then, if we want to find the fifth term of this sequence. \( a_5 = \frac{1}{3} \), by the definition of the functional notation.
3.1. SEQUENCES AND SERIES

It is clear that if we know the general term of a sequence, then we can get any term of the sequence. Because of that, we express sequences just using the \( n^{th} \) term as in the following notation.

\[
\{a_n\}_{n=1}^{\infty}
\]

This notation means, “the set of all \( a_n \), where \( n \) varies over natural numbers”. That is,

\[
\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}
\]

**Exercise.** Identify the first five terms of the given sequence.

1. \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \)
2. \( \left\{ \frac{k^2}{k+2} \right\}_{k=1}^{\infty} \)
3. \( \left\{ \frac{n!}{n+3} \right\}_{n=1}^{\infty} \)

*Here, \( n! \) stands for the number \( n(n-1)(n-2)\cdots3\cdot2\cdot1 \). For example, \( 5! = 5\cdot4\cdot3\cdot2\cdot1 \).*

Consider a sequence \( \{a_n\}_{n=1}^{\infty} \). The sum of the first \( n \) terms of the sequence is called the \( n^{th} \) partial sum of the sequence. We will usually use \( S_n \) to represent the \( n^{th} \) partial sum of a sequence. That is,

\[
S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n.
\]

If we can add all elements of a sequence, then we call it a series. For example,

\[
a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n + \cdots
\]

is a series. However, this notation is not that great as the last three dots (\( \cdots \)), which usually stand for “and so on” now stand for “and so on without bounds”. We can use the following superior notation to represent a series.

\[
\sum_{n=1}^{\infty} a_n
\]

This notation stands for “the sum of all \( a_n \), where \( n \) varies over natural numbers”.

This notation also gives us another way to represent the \( n^{th} \) partial sum of a sequence.

\[
\sum_{k=1}^{n} a_k
\]

The above notation stands for “the sum of all \( a_k \), where \( k \) varies over natural numbers from 1 to \( n \)”. The \( n^{th} \) partial sum of a sequence is occasionally known as a finite series.
Exercise. Identify the first five terms of the sequence

1. Find the 5th partial sum of the sequence \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \).

2. Find the 5th partial sum of the sequence \( \left\{ \frac{k^2}{k+2} \right\}_{k=1}^{\infty} \).

3. Find the 5th partial sum of the sequence \( \left\{ \frac{n!}{n+3} \right\}_{n=1}^{\infty} \).

4. Find \( \sum_{k=1}^{5} \frac{k+1}{k+2} \).

3.1.1 Arithmetic Sequences and Series

Consider the following sequence.

\[
\left\{ 1 + \frac{1}{2}(n - 1) \right\}_{n=1}^{\infty}
\]

The first seven terms of this sequence are \( \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \ldots \} \).

Notice that the difference between any two consecutive terms of the sequence is \( \frac{1}{2} \).
3.1. SEQUENCES AND SERIES

\[ a_1 = 1 + \frac{1}{2}(1 - 1) = 1 \]
\[ a_2 = 1 + \frac{1}{2}(2 - 1) = \frac{3}{2} = 1 + \frac{1}{2} \]
\[ a_3 = 1 + \frac{1}{2}(3 - 1) = 2 = \frac{3}{2} + \frac{1}{2} \]
\[ a_4 = 1 + \frac{1}{2}(4 - 1) = \frac{5}{2} = 2 + \frac{1}{2} \]
\[ a_5 = 1 + \frac{1}{2}(5 - 1) = 3 = \frac{5}{2} + \frac{1}{2} \]
\[ a_6 = 1 + \frac{1}{2}(6 - 1) = \frac{7}{2} = 3 + \frac{1}{2} \]
\[ a_7 = 1 + \frac{1}{2}(7 - 1) = 4 = \frac{7}{2} + \frac{1}{2} \]
\[ \vdots \]
\[ a_k = 1 + \frac{1}{2}(k - 1) \]
\[ a_{k+1} = 1 + \frac{1}{2}((k + 1) - 1) = 1 + \frac{1}{2}k = a_k + \frac{1}{2}, \]
where \( k \) is an arbitrary positive integer.

A sequence with this property, that is, the \textit{difference between any two consecutive terms is a constant}, is called an \textit{arithmetic sequence}. The difference between any two consecutive terms of an arithmetic sequence is called the \textit{common difference}.

\begin{tabular}{|c|}
\hline
\textbf{Theorem.} Let \( a \) be the first term of an arithmetic sequence and let \( d \) be the common difference. Then the general term of the sequence is \( a_n = a + (n - 1)d \). \\
\hline
\end{tabular}

\textbf{Exercise.} Prove the previous theorem.

If we denote \( a \) as the first term of an arithmetic sequence\(^1\) and \( d \) as the common difference, then we can denote \textit{every} element of the arithmetic sequence in terms of \( a \) and \( d \) as shown below:

\[ \{a, (a + d), (a + 2d), (a + 3d), \ldots, (a + (n - 1)d), \ldots\} \]

\textbf{Example.} Show that \( \{15 - \frac{4}{3}n\}_{n=1}^{\infty} \) is an arithmetic sequence.

\(^1\)It is a convention to write \( a_1 \) as just \( a \) for arithmetic sequences and soon to be introduced geometric sequences.
Solution. By definition, if the difference between any two consecutive terms is a constant, then the sequence is an arithmetic sequence. Therefore, our goal is to show that the difference between any two consecutive terms of the given sequence is a constant.

On the face of it, this is an impossible task. There are infinitely many natural numbers and therefore, there are infinitely many pairs of consecutive terms to check. The mathematical solution to this conundrum is the following. We will show that the difference between \( a_{k+1} \) and \( a_k \) is a constant, for an arbitrary natural number \( k \). If we can do that, then since \( k \) is arbitrary, we have shown that the difference between any two consecutive terms of the sequence is a constant. This is a standard mathematical technique when you are left to check infinitely many cases.

Let \( k \) be an arbitrary natural number. Then \( a_k = 15 - \frac{4}{3}k \) and \( a_{k+1} = 15 - \frac{4}{3}(k+1) \).

Therefore,

\[
a_{k+1} - a_k = \left( 15 - \frac{4}{3}(k+1) \right) - \left( 15 - \frac{4}{3}k \right)
= 15 - \frac{4}{3}k - \frac{4}{3} - 15 + \frac{4}{3}k
= \frac{4}{3}
\]

Since \( k \) is arbitrary, the difference between any two consecutive terms of the given sequence is the constant \(-\frac{4}{3}\). Therefore, the given sequence is an arithmetic sequence.

Let \( S_n \) be the \( n^{th} \) partial sum of an arithmetic sequence with the first term \( a \) and the common difference \( d \). Then

\[
S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-3)d) + (a+(n-2)d) + (a+(n-1)d).
\]

For the following argument, let us look at the first term \( a_1 \), that is \( a \), as \( a + 0d \). In other words, the first term is a sum of “an \( a \)” and “no \( d \)” terms. The \( n^{th} \) term is \( a + (n-1)d \). That is, the \( n^{th} \) term is a sum of “an \( a \)” and “\( (n-1) \) number of \( d \) terms”. If we add the first term and the last term together, that is, \( a_1 + a_n \), then we get \( 2a + (n-1)d \). Now look at \( a_2 + a_{n-1} \): \( a_2 \) has one more \( d \) than \( a_1 \), and \( a_{n-1} \) has one less \( d \) that \( a_n \). Therefore, \( a_2 + a_{n-1} \) has the same number of \( d \) terms as \( a_1 + a_n \). Therefore, \( a_2 + a_{n-1} = 2a + (n-1)d \) as well. If we look at \( a_3 + a_{n-2} \), then \( a_3 \) has two more \( d \) terms than \( a_1 \), and \( a_{n-2} \) has two less \( d \) terms that \( a_n \). Therefore, \( a_3 + a_{n-2} \) has the same number of \( d \) terms as \( a_1 + a_n \). Therefore, \( a_3 + a_{n-2} \) is also equal to \( 2a + (n-1)d \). This kind of thinking leads us to the next theorem.
Theorem. If \( \{a_n\}_{n=1}^{\infty} \) is an arithmetic sequence with \( a_1 = a \) and the common difference \( d \), then the \( n^{th} \) partial sum, \( S_n \), is given by the following formula.

\[
S_n = \frac{n}{2}[2a + (n-1)d]
\]

Proof. Case 1: Suppose \( n \) is even. Then \( \frac{n}{2} \) is a natural number, since \( n \) is divisible by 2. There are two middle terms and the middle two terms are \( a_{n/2} \) and \( a_{n/2+1} \).

\[
S_n = a_1 + a_2 + a_3 + \cdots + a_{n/2} + a_{n/2+1} + \cdots + a_n.
\]

This sum can be written as:

\[
S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots + (a_{n/2} + a_{n/2+1}).
\]

Which is the same as:

\[
S_n = [2a + (n-1)d] + [2a + (n-1)d] + [2a + (n-1)d] + \cdots + [2a + (n-1)d].
\]

The term \( [2a + (n-1)d] \) in \( S_n \) repeats \( \frac{n}{2} \) in \( S - n \). Therefore,

\[
S_n = \frac{n}{2}[2a + (n-1)d].
\]

Case 2: Now suppose \( n \) is odd. Then \( n+1 \) is even and \( \frac{n+1}{2} \) is a natural number since \( n+1 \) is divisible by 2. In this case, there is only one middle term, and the middle term is \( a_{(n+1)/2} \).

According to the general term of an arithmetic sequence, \( a_{(n+1)/2} = a + ((n+1)/2 - 1)d \).

\[
S_n = a_1 + a_2 + a_3 + \cdots + a_{(n-1)/2} + a_{(n+1)/2} + a_{(n+3)/2} + \cdots + a_{n-2} + a_{n-1} + a_n.
\]

This sum can be written as:

\[
S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots + (a_{(n-1)/2} + a_{(n+3)/2}) + a_{(n+1)/2}.
\]

Which is the same as:

\[
S_n = \frac{n-1}{2}[2a + (n-1)d] + a_{(n+1)/2}.
\]

By identifying the \( a_{(n+1)/2} \) term we get:

\[
S_n = \frac{n-1}{2}[2a + (n-1)d] + \left[a + \left(\frac{n+1}{2} - 1\right)d\right].
\]
Which is the same as:

\[ S_n = \frac{n-1}{2} [2a + (n-1)d] + \left[ a + \left( \frac{n-1}{2} \right) d \right]. \]

That is,

\[ S_n = \frac{n-1}{2} [2a + (n-1)d] + \frac{1}{2}[2a + (n-1)d]. \]

By factoring out the common term, we get:

\[ S_n = \left( \frac{n-1}{2} + \frac{1}{2} \right) [2a + (n-1)d]. \]

Which is the same as:

\[ S_n = \frac{n}{2} [2a + (n-1)d]. \]

\[ \square \]

**Example.** Find \( \sum_{n=1}^{100} 2n \), if possible.

**Solution.** We can restate this problem as follows. Find the 100\textsuperscript{th} partial sum of the sequence \( \{2n\}_{n=1}^{\infty} \). If the sequence \( \{2n\}_{n=1}^{\infty} \) is arithmetic, then we can use the previous theorem to answer the given question. Therefore, our first order of business is to check and see if the sequence is arithmetic. For an arbitrary natural number \( k \),

\[ a_{k+1} - a_k = 2(k + 1) - 2k = 2k + 2 - 2k = 2 \]

Since \( k \) is arbitrary, the difference between any two consecutive terms of the sequence \( \{2n\}_{n=1}^{\infty} \) is 2. Therefore, \( \{2n\}_{n=1}^{\infty} \) is arithmetic, with the first term 2 and the common difference 2. Then

\[ S_{100} = \frac{100}{2}[2(2) + (100 - 1)(2)] \]

\[ = 50[4 + (99)(2)] \]

\[ = 10,100. \]
3.1.2 Geometric Sequences and Series

The following definition and the theorem that you may have learned in high school are extremely useful in this section.

**Definition.** Let $a \neq 0$ be a real number and $n$ be a positive integer. Then

1. $a^0 = 1$.
2. $a^1 = a$
3. $a^{-n} = \frac{1}{a^n}$

**Integer Powers Theorem.**\(^2\) Suppose $a \neq 0$ and $b \neq 0$ are real numbers and $m$ and $n$ are integers. Then

1. $a^n a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$
3. $\frac{a^n}{a^m} = a^{n-m}$
4. $(ab)^n = a^n b^n$
5. $(\frac{a}{b})^n = a^n b^{-n}$

Consider the following sequence.

\[
\left\{ 3 \left( \frac{1}{2} \right)^{n-1} \right\}_{n=1}^{\infty}
\]

The first five terms of this sequence are \{3, $\frac{3}{2}$, $\frac{3}{2^2}$, $\frac{3}{2^3}$, $\frac{3}{2^4}$ \ldots \}.

\(^2\)You may have proved this theorem for rational powers in high school.
Notice that the ratio between any two consecutive terms of the sequence is \( \frac{1}{2} \).

\[
\begin{align*}
a_1 &= 3 \left( \frac{1}{2} \right)^{1-1} = 3 \\
a_2 &= 3 \left( \frac{1}{2} \right)^{2-1} = \frac{3}{2} = a_1 \left( \frac{1}{2} \right) \\
a_3 &= 3 \left( \frac{1}{2} \right)^{3-1} = \frac{3}{2^2} = a_2 \left( \frac{1}{2} \right) \\
a_4 &= 3 \left( \frac{1}{2} \right)^{4-1} = \frac{3}{2^3} = a_3 \left( \frac{1}{2} \right) \\
a_5 &= 3 \left( \frac{1}{2} \right)^{5-1} = \frac{3}{2^4} = a_4 \left( \frac{1}{2} \right) \\
\vdots \\
a_k &= 3 \left( \frac{1}{2} \right)^{k-1} \\
a_{k+1} &= 3 \left( \frac{1}{2} \right)^{k+1-1} = \frac{3}{2^k} = a_k \left( \frac{1}{2} \right)
\end{align*}
\]

where \( k \) is an arbitrary positive integer.

A sequence with this property, that is, the ratio between any two consecutive terms is a constant, is called a geometric sequence. The ratio between any two consecutive terms of a geometric sequence is called the common ratio—textbf.

**Theorem.** Let the first term of a geometric sequence be \( a \) and let the common ratio be \( r \). Then the general term of the sequence is \( a_n = ar^{n-1} \).

**Exercise.** Prove the previous theorem.

Let \( a \) denote the first term of a geometric sequence and let \( r \) denote the common ratio. Then we can denote every element of the geometric sequence in terms of \( a \) and \( r \) as shown
below:
\[ \{a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots\} \]

**Example.** Show that \( \{15 \left(\frac{4}{3}\right)^n\}_{n=1}^{\infty} \) is a geometric sequence.

**Solution.** By definition, if the ratio between any two consecutive terms is a constant, then the sequence is a geometric sequence. Therefore, our goal is to show the ratio between any two consecutive terms of the given sequence is a constant.

Let \( k \) be an arbitrary natural number. Then
\[
\frac{a_{k+1}}{a_k} = \frac{15 \left(\frac{4}{3}\right)^{k+1}}{15 \left(\frac{4}{3}\right)^k} = \frac{4}{3}
\]

Since \( k \) is arbitrary, the ratio between any two consecutive terms of the given sequence is the constant \( \frac{4}{3} \). Therefore, the given sequence is a geometric sequence.

**Finite Geometric Series Theorem.** Given a geometric sequence \( \{ar^{n-1}\}_{n=1}^{\infty} \), where \( a \) is the first term and \( r \) is the common ratio, then the \( n^{th} \) partial sum of \( \{ar^{n-1}\}_{n=1}^{\infty} \) is
\[
S_n = \frac{a(1 - r^n)}{1 - r}.
\]

**Proof.** Let \( S_n \) be the \( n^{th} \) partial sum of a geometric sequence with the first term \( a \) and the common ratio \( r \). Then
\[
S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}.
\]

Notice that
\[
rS_n = ar + ar^2 + ar^3 + ar^4 \cdots + ar^{n-2} + ar^{n-1} + ar^n.
\]

Subtracting these equations gives
\[
S_n - rS_n = a - ar^n
\]

Solving for \( S_n \) gives the formula for the \( n^{th} \) partial sum of a geometric sequence.
That is,
\[ rS_n - ar^n = ar + ar^2 + ar^3 + ar^4 \cdots + ar^{n-2} + ar^{n-1}. \]

That is,
\[ rS_n - ar^n = -a + a + ar + ar^2 + ar^3 + ar^4 \cdots + ar^{n-2} + ar^{n-1}. \]

That is,
\[ rS_n - ar^n = -a + S_n. \]

Now, this is a linear equation in \( S_n \). By solving this linear equation for \( S_n \), we get:
\[ S_n = \frac{a(1 - r^n)}{1 - r}. \]

Example. Find \( \sum_{n=1}^{20} \left( \frac{1}{2} \right)^n \), if possible.

Solution. For an arbitrary natural number \( k \),
\[
\frac{a_{k+1}}{a_k} = \left( \frac{1}{2} \right)^{k+1} \left( \frac{1}{2} \right)^k = \left( \frac{1}{2} \right)^{k+1-k} = \left( \frac{1}{2} \right) = \frac{1}{2}
\]

Since \( k \) is arbitrary, the ratio between any two consecutive terms of the sequence \( \left\{ \left( \frac{1}{2} \right)^n \right\}_{n=1}^{\infty} \) is \( \frac{1}{2} \). Therefore, \( \left\{ \left( \frac{1}{2} \right)^n \right\}_{n=1}^{\infty} \) is geometric, with the first term \( \frac{1}{2} \) and the common ratio \( \frac{1}{2} \). If we write the general term in the form \( ar^{n-1} \), then the geometric sequence is \( \left\{ \frac{1}{2} \left( \frac{1}{2} \right)^{n-1} \right\}_{n=1}^{\infty} \).

Now we can calculate \( S_{20} \) by using the Finite Geometric Series Theorem.
\[
S_{20} = \frac{\frac{1}{2} \left( 1 - \left( \frac{1}{2} \right)^{20} \right)}{1 - \frac{1}{2}} = \frac{\frac{1}{2} \left( 1 - \frac{1}{2^{20}} \right)}{\frac{1}{2}}
\]
3.1. SEQUENCES AND SERIES

\[ S_{20} = 1 - \frac{1}{2^{20}} \]

\[ = 1 - \frac{1}{1048576} \]

\[ = \frac{1048576}{1048576} - 1 \]

\[ = \frac{1048575}{1048576} \]

In the previous example, \( S_{20} \), is very close to 1, but less than 1. Would the \( n^{th} \) partial sum be more than 1 for some high natural number \( n \)? We can do a calculator experiment, within the limitations of TI-83 calculator. (The largest exact value for \( 2^n \) that can be obtained in a TI-83 calculator is when \( n = 33 \).)

**Example.** Find \( \sum_{n=1}^{33} \left( \frac{1}{2} \right)^n \), if possible.

**Solution.**

\[ S_{33} = \frac{\frac{1}{2} \left( 1 - \left( \frac{1}{2} \right)^{33} \right)}{1 - \frac{1}{2}} \]

\[ = \frac{\frac{1}{2} \left( 1 - \frac{1}{2^{33}} \right)}{\frac{1}{2}} \]

\[ = 1 - \frac{1}{2^{33}} \]

\[ = 1 - \frac{1}{8589934592} \]

\[ = \frac{8589934592 - 1}{8589934592} \]

\[ = \frac{8589934591}{8589934592} \]

The 33\textsuperscript{rd} partial sum is even closer to 1 but still less than 1. Since we have reached the limits of calculators, we need to look for a purely mathematical way of moving forward.

If we can show that, \( \left( \frac{1}{2} \right)^n \rightarrow 0 \), for \( n \rightarrow \infty \), then it follows that, for larger values of \( n \), \( S_n \approx 1 \), by the Finite Geometric Series Sum Theorem.
The following theorem is beyond the scope of Precalculus. (This is the second such theorem that we have encountered so far.) However, it is not hard to imagine its truth.

**The Archimedean Property Theorem.** Suppose \( R \) is a positive real number. Then there is a natural number \( N \) so that \( N > R \).

That is, no matter how big the real number \( R \) is, we can *always* find a natural number bigger than \( R \).

The following theorem is also needed; however, we cannot prove it yet. We will prove this theorem when we learn the method, “Proof by Mathematical Induction”, in the next lesson.

**Theorem** (Theorem 5). Suppose \( n \) is any natural number. Then \( 2^n > n \).

**Theorem.** Suppose \( r = \frac{1}{2} \). Then, for \( n \to \infty \), \( r^n \to 0 \).

Notation: We will use \( \implies \) to replace the English word “implies”.

**Proof.** Let \( r = \frac{1}{2} \). Clearly, \( r < 1 \). We will use Theorem 3 repeatedly.

\[
r < 1 \implies r^2 < r.
\]
\[
r^2 < r \implies r^3 < r^2.
\]
\[
r^3 < r^2 \implies r^4 < r^3.
\]
\[
\vdots
\]
\[
r^k < r^{k-1} \implies r^{k+1} < r^k.
\]
\[
\vdots
\]
That is, if $0 < r < 1$, then $0 < \cdots r^{k+1} < r^k < \cdots < r^4 < r^3 < r^2 < r < 1$, where, $k$ is a sufficiently large arbitrary natural number. (In this case, $k > 4$ by choice.)

That is, as “$n$ increases” the value of “$r^n$ decreases” while remaining positive. Could $r^n$ be always bigger than some small, but positive number $\epsilon$, for any $n$? If so, then for $n \to \infty$, $r^n \to \epsilon$. Our gut feeling is that this is not the case. But how can we be sure it is not the case? One mathematical method that we can use is to assume a given statement is true and then look for a contradiction that will show that the assumption is false. This type of proof is known as “proof by contradiction”. (We have used this method when solving equations without mentioning the name “proof by contradiction”.)

Suppose that there is a positive number $\epsilon$ so that $\epsilon < r^n < 1$, for all natural numbers $n$. Then $\frac{1}{\epsilon} > 1$. By Archimedean Property Theorem, there is a natural number $N$ so that $N > \frac{1}{\epsilon}$. By Theorem 5, $2^N > N$. That is, $2^N = N > \frac{1}{\epsilon}$ Then $\frac{1}{2^N} < \frac{1}{N} < \epsilon$. That is, $\left(\frac{1}{2}\right)^N < \frac{1}{N} < \epsilon$. That is, $r^N < \frac{1}{N} < \epsilon$. This contradicts our assumption that $r^n > \epsilon$, for all natural numbers $n$. Therefore, our assumption is false, and $r^n \to 0$ for $n \to \infty$.

The following general theorem is also true. But the proof is hard (as you may have guessed from the special case of this theorem that we just proved).

**Theorem** (Theorem 6). Suppose $|r| < 1$. Then, for $n \to \infty$, $r^n \to 0$.

We will use the symbol $S_\infty$ to represent $\sum_{n=1}^{\infty} a_n$. That is, $S_\infty$ stands for the infinite series $\sum_{n=1}^{\infty} a_n$.

**Infinite Geometric Series Theorem.** Consider the geometric sequence $\{ar^{n-1}\}_{n=1}^{\infty}$, where $a$ is the first term and $r$ is the common ratio. Suppose $|r| < 1$. Then $S_\infty = \frac{a}{1-r}$.

**Proof.** The proof of this theorem follows from the Finite Geometric Series Theorem and Theorem 6.

**Example.** Find $S_\infty$ for the sequence $\{(\frac{8}{9})^n\}$, if possible.
Solution. For an arbitrary natural number \( k \),

\[
\frac{a_{k+1}}{a_k} = \left(\frac{8}{9}\right)^{k+1} \left(\frac{8}{9}\right)^k = \left(\frac{8}{9}\right)^{k+1-k} = \frac{8}{9}
\]

Since \( k \) is arbitrary, the ratio between any two consecutive terms of the sequence \( \{(\frac{8}{9})^n\}_{n=1}^{\infty} \) is \( \frac{8}{9} \). Therefore, \( \{(\frac{8}{9})^n\}_{n=1}^{\infty} \) is geometric, with the first term \( \frac{8}{9} \) and the common ratio \( \frac{8}{9} \).

Clearly, \( |\frac{8}{9}| < 1 \). Therefore, by the Infinite Geometric Series Theorem,

\[
S_\infty = \frac{\left(\frac{8}{9}\right)}{1 - \frac{8}{9}} = \frac{\frac{8}{9}}{\frac{1}{9}} = 8 \quad \square
\]

3.2 Mathematical Induction

This is a very important method of proving theorems with the following characteristics.

1. The theorem contains infinitely many countable statements to prove.
2. There is a first statement.
3. For any statement, there is a next statement.

Clearly, the proof by mathematical induction is intimately related to the natural numbers; as there are infinitely many (countable) natural numbers; there is a first natural number; and for any natural number, there is a next natural number. The proof by mathematical induction is a two-step process. The first step, known as the “first step” (no surprise here!) is to prove the first statement. The second step, known as the “inductive step” is to assume that the \( k^{th} \) statement is true, for some natural number \( k \), and then prove that the statement next to the \( k^{th} \) statement is also true. If you are successful in completing
the first step and the inductive step, then we say that, by mathematical induction, all statements must be true.

The argument goes as follows. We have shown that the first statement is true. Then by the inductive step, the next statement, that is, the second statement is also true. Since the second statement is true, then by the inductive step, the statement following the second statement, that is, the third statement, is also true. By continuing in this process, it should be clear that you can prove the \( n^{th} \) statement, for any natural number \( n \).

**Example.** *Prove that \( 2^n > n \), for all natural numbers \( n \), by using mathematical induction.*

**Proof.** First notice that there are infinitely many countable statements to prove in the exercise. Namely:

1. Show that \( 2^1 > 1 \).
2. Show that \( 2^2 > 2 \).
3. Show that \( 2^3 > 3 \).

\[
\vdots
\]

and ad infinitum. (That is, again and again in the same way, forever.)

Clearly, this is an impossible task — if you insist on proving each statement. But, fortunately, this theorem satisfies the three requirements that we need to use the proof by mathematical induction. That is, (1) there are infinitely many statements; (2) there is a first statement; and (3) for each statement there is a next statement. Therefore, we will use mathematical induction to prove this theorem.

**First step:** Show that the first statement of the theorem is true. That is, show that

\[ 2^1 > 1 \text{ is true.} \]

Clearly, \( 2 > 1 \). Therefore, the first statement of the theorem is true.

**Inductive step:** Assume that the \( k^{th} \) statement of the theorem is true, for some arbitrary natural number \( k \). That is, assume:

\[ 2^k > k \text{ is true. —— (A)} \]

and then show that

\[ 2^{k+1} > k + 1 \text{ is also true. —— (B)} \]
Let us attend to the task at hand now.

The left side of (B) \( = 2^{k+1} \)
\[ = 2(2^k) \]
\[ > 2k, \text{ by (A).} \]
\[ = k + k \]
\[ \geq k + 1 \text{ because, } k \geq 1. \]

That is, \( 2^{k+1} > k + 1 \) is true.

Thus, we showed that, if the \( k^{th} \) statement of the theorem is true, for some arbitrary natural number \( k \), then the \( k + 1^{st} \) statement of the theorem is also true.

Therefore, by mathematical induction, the theorem must be true for any natural number \( n \). \( \square \)

**Example.** Prove that \( x - 1 \) is a factor of \( x^n - 1 \) for all natural numbers \( n \), by using mathematical induction.

**Proof.** Clearly, there are infinitely many countable statements to prove; there is a first statement; and for each statement, there is a next statement. Therefore, we can use mathematical induction to prove this theorem.

**First step:** Show that the first statement of the theorem is true. That is, show that \( x - 1 \) is a factor of \( x^1 - 1 \). This statement is clearly true, since \( (1)(x - 1) = x - 1 \).

**Inductive step:** Assume that the \( k^{th} \) statement of the theorem is true, for some arbitrary natural number \( k \). That is, assume that:

\[ x - 1 \text{ is a factor of } x^k - 1 \text{ is true.} \quad (A) \]

and show that

\[ x - 1 \text{ is a factor of } x^{k+1} - 1 \text{ is also true.} \quad (B) \]

Now, let us prove (B). First, notice that there is a polynomial \( p(x) \) so that \( x^k - 1 = (x - 1)p(x) \), by (A).

\[ x^{k+1} - 1 = x^{k+1} - x + x - 1 \]
\[ = x(x^k - 1) + (x - 1) \]
That is, if \((x - 1)\) is a factor of \(x^k - 1\), then \((x - 1)\) is a factor of \(x^{k+1} - 1\).

Therefore, by mathematical induction, the theorem is true for any natural number \(n\).

**Example.** Prove that

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}
\]

for all natural numbers \(n\), by using mathematical induction.

**Proof.** Notice first that the left side of the given equation is the \(n^{th}\) partial sum of the sequence \(\left\{ \frac{1}{n(n + 1)} \right\}_{n=1}^{\infty}\). Let \(S_n\) be the \(n^{th}\) partial sum of the sequence \(\left\{ \frac{1}{n(n + 1)} \right\}_{n=1}^{\infty}\), for any natural number \(n\). Then, the task of the theorem is to show that \(S_n = \frac{n}{n+1}\), for all natural numbers \(n\).

**First step:** Show that the first statement of the theorem is true. That is, show that

\[
S_1 = \frac{1}{1+1} = \frac{1}{2} \text{ is true.}
\]

Therefore, \(S_1 = \frac{1}{1+1}\) true. That is, the first statement is true.

**Inductive step:** Assume that the \(k^{th}\) statement of the theorem is true, for some arbitrary natural number \(k\). That is, assume

\[
S_k = \frac{k}{k+1} \text{ is true,}
\]

That is, assume

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{k(k + 1)} = \frac{k}{k+1} \text{ is true, —— (A)}
\]

and show that

\[
S_{k+1} = \frac{k + 1}{(k + 1) + 1} \text{ is also true,}
\]
which is to say, show that
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}
\]
is also true, —— (B)

Now, let us prove (B).

The left side of (B) =
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}
\]
\[= S_k + \frac{1}{(k+1)(k+2)} \]
\[= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ by (A).} \]
\[= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}
\]
\[= \frac{k^2 + 2k + 1}{(k+1)(k+2)}
\]
\[= \frac{(k+1)^2}{(k+1)(k+2)}, \text{ by the Three Identities Theorem.} \]
\[= \frac{k+1}{k+2}, \text{ since } k+1 \neq 0. \]
\[= \text{The right side of (B)}
\]

Therefore, if the \(k^{th}\) statement is true, for some arbitrary natural number \(k\), then the \((k+1)^{st}\) statement is also true.

Therefore, by mathematical induction, the theorem is true for any natural number \(n\).

**Exercise.** Let \(S_n\) be the \(n^{th}\) partial sum of the sequence \(\{n\}_{n=1}^{\infty}\). Show that

\[S_1 + S_2 + S_3 + \cdots + S_n = \frac{1}{6}n(n+1)(n+2),\]

for any natural number \(n\).
Chapter 4

Partial Fraction Decomposition

4.1 Solving Systems of Linear Equations (Review)

You may have learned how to solve systems of linear equations in high school. This section is meant as a review.

Example. Find the solution of the system

\[
\begin{align*}
3A + 4B - C &= 5 \\
-4A + 7C &= -4 \\
A - 2B + 4C &= 1,
\end{align*}
\]

where \(A, B,\) and \(C\) are real numbers.

Solution. We will assume that all three equations are true for some real numbers \(A, B,\) and \(C.\) Then by the Four Properties Theorem, the following statements are true.

(i) If we multiply one equation by a non-zero number, then the new system of equations is still true.

(ii) If we add one equation to another equation, then the new system of equations is still true.

We will perform a series of actions of types (i) and (ii), until we get a system of three
equations

\[ A = a \]
\[ B = b \]
\[ C = c \]  \hspace{1cm} (II)

which are all true by design, if our original assumption is true. At this point we will check and see if the solutions actually satisfy the original system of equations. If so, then, the triple \((a, b, c)\) is called the solution of the system.

The following is one way to get from system \((I)\) to system \((II)\).

Multiply the third equation of the system \((I)\) by 4.

\[
3A + 4B - C = 5 \\
-4A + 7C = -4 \\
4A - 8B + 16C = 4
\] \hspace{1cm} (4.1)

Add the second equation in the system \((1)\) to the third equation in the system \((1)\).

\[
3A + 4B - C = 5 \\
-4A + 7C = -4 \\
-8B + 23C = 0
\] \hspace{1cm} (4.2)

Multiply the first equation of the system \((2)\) by 4, and multiply the second equation of the system \((2)\) by 3.

\[
12A + 16B - 4C = 20 \\
-12A + 21C = -12 \\
-8B + 23C = 0
\] \hspace{1cm} (4.3)
Add the first equation in the system (3) to the second equation in the system (3).

\[
12A + 16B - 4C = 20 \\
16B + 17C = 8 \\
-8B + 23C = 0
\]  

Multiply the third equation in the system (4) by 2.

\[
12A + 16B - 4C = 20 \\
16B + 17C = 8 \\
-16B + 46C = 0
\]  

Add the second equation in the system (5) to the third equation in the system (5).

\[
12A + 16B - 4C = 20 \\
16B + 17C = 8 \\
63C = 8
\]  

Multiply the second equation of the system (6) by 63, and multiply the third equation of the system (6) by -17.

\[
12A + 16B - 4C = 20 \\
1008B + 1071C = 504 \\
-1071C = -136
\]  

Add the third equation in the system (7) to the second equation in the system (7).

\[
12A + 16B - 4C = 20 \\
1008B = 368 \\
-1071C = -136
\]
Multiply the second equation in the system (8) by \( \frac{1}{1008} \), and multiply the third equation in the system (8) by \( -\frac{1}{1071} \), and reduce the fractions.

\[
12A + 16B - 4C = 20
\]

\[
B = \frac{23}{63}
\]

\[
C = \frac{8}{63}
\]

(Multiply the second equation in the system (8) by \( -\frac{1}{1071} \), and reduce the fractions.

\[
12A - 4C = \frac{892}{63}
\]

\[
B = \frac{23}{63}
\]

\[
C = \frac{8}{63}
\]

(Multiply the first equation in the system (9) by \( \frac{1}{4} \).

\[
3A - C = \frac{223}{63}
\]

\[
B = \frac{23}{63}
\]

\[
C = \frac{8}{63}
\]

(Add the third equation in the system (11) to the first equation in the system (11).

\[
3A = \frac{11}{3}
\]

\[
B = \frac{23}{63}
\]

\[
C = \frac{8}{63}
\]

(Multiply the first equation in the system (12) by \( \frac{1}{3} \).

\[
A = \frac{11}{9}
\]

\[
B = \frac{23}{63}
\]

\[
C = \frac{8}{63}
\]
After checking with the system (I), \( A = \frac{11}{9} \), \( B = \frac{23}{63} \), and \( C = \frac{8}{63} \) indeed satisfy the three equations in system (I). Therefore, the solution of the system is \( \left( \frac{11}{9}, \frac{23}{63}, \frac{8}{63} \right) \).

Let us look at the system (I) in the following way.

\[
\begin{pmatrix}
\begin{array}{cccc}
3 & 4 & -1 & 5 \\
-4 & 0 & 7 & -4 \\
1 & -2 & 4 & 1 \\
\end{array}
\end{pmatrix}
\]  

(III)

The first row represents the first equation of the system (I): the first entry is the coefficient of \( A \); the second entry is the coefficient of \( B \); the third entry is the coefficient of \( C \); and the last entry is the right side of the first equation. The second row represents the second equation of the system (I); and the third row represents the third equation of the system (I). The system in (III) is called the matrix form of the system in (I).

If we write the matrix form of the system in (II), then we get:

\[
\begin{pmatrix}
\begin{array}{cccc}
1 & 0 & 0 & \frac{11}{9} \\
0 & 1 & 0 & \frac{23}{63} \\
0 & 0 & 1 & \frac{8}{63} \\
\end{array}
\end{pmatrix}
\]  

(IV)

The matrix (IV) is called the reduced raw echelon form (rref) of the matrix (III).

If you have noticed the algorithmic nature of the process of going from matrix (III) to matrix (IV), then it may have occurred to you that this process can be programmed into a calculator to speedily find the solution of a system of linear equations.

In fact, there is a command in TI-83 called “rref” that is the name of the program to find the reduced raw echelon form of a matrix in TI-83.

**Exercise.** Suppose \( A \), \( B \), and \( C \) are real numbers. Find the solution of the system

\[
\begin{align*}
-A + 2B - 3C &= 7 \\
4A - B + C &= 4 \\
A - 2B + 4C &= 6
\end{align*}
\]
by using the “rref” command in TI-83. Write your solution as a triple of fractions. That is, decimal approximations are unacceptable.

4.2 Partial Fraction Decomposition: Linear Factors

Consider the following two examples of the addition of two rational expressions. You may have learned in grades 4 and 5, how to add fractions. Then you may have learned in high school that the addition of fractional expression are similar to the addition of fractions.

Example. Find
\[
\frac{3}{x - 13} + \frac{14}{x + 14}
\]

The basic idea behind the addition of fractions is that the fractions can be put on an “equal footing”. Once that is done the addition of fractions is reduced to an addition of positive integers that you may have learned in grade 1 and 2. We will do nothing different here.

\[
\frac{3}{x - 13} + \frac{14}{x + 14} = \frac{3(x + 14)}{(x - 13)(x + 14)} + \frac{14(x - 13)}{(x - 13)(x + 14)}
\]
\[
= \frac{3(x + 14) + 14(x - 13)}{(x - 13)(x + 14)}
\]
\[
= \frac{17x - 140}{(x - 13)(x + 14)} \quad \square
\]

Example. Find
\[
\frac{3}{x - 3} + \frac{4}{(x - 3)^2} + \frac{5}{(x - 3)^3}
\]
Recall the use of the distributive property with polynomials and the factoring of polynomials. For example, we can multiply \((x - 2)(x + 3)\) by using the distributive property. That is, \((x - 2)(x + 3) = x^2 + x - 6\). Now if we want to factor \(x^2 + x - 6\), we have to devise a method to undo what distributive property did to \((x - 2)(x + 3)\). This should be familiar to you.

We will ask similar questions now.

1. Can you write \(\frac{17x - 140}{(x - 13)(x + 14)}\) as a sum of fractional expressions?

2. Can you write \(\frac{3x^2 - 14x + 20}{(x - 3)^3}\) as a sum of fractional expressions?

If we have not seen the two previous examples, then at the moment we do not know what to do. Writing a given fractional expression as a sum of fractional expressions is known as partial fraction decomposition (PFD).

In the first example, we say the linear factors \((x - 13)\) and \((x + 14)\) are non-repeating factors. In the second example, we say \((x - 3)\) is a repeating linear factor. In fact, \((x - 3)\) is repeating 3 times.

The following useful theorem is the third theorem that we have encountered that we cannot prove in this class.
**PFD with Linear Factors Theorem.** Suppose \( f(x) = \frac{p(x)}{q(x)} \) is a rational function, where, \( p(x) \) and \( q(x) \) are polynomials and the degree of \( p(x) \) < the degree of \( q(x) \). Suppose \( q(x) \) can be completely factored into linear factors.

1. For each non-repeating linear factor of \( q(x) \), write a fractional expression of the form \((A_k)/(\text{linear factor})\), where \( A_k \), etc., are constants.
2. For each repeating linear factor that repeats \( n \) times, write \( n \) factors of the form \((B_1)/(\text{linear factor}), (B_2)/(\text{linear factor})^2, \ldots (B_n)/(\text{linear factor})^n\), where \( B_1, B_2, \ldots, B_n \), etc., are constants.

Then \( f(x) = \text{sum of all fractional expressions in (1) and (2), and all those constants in the numerators of fractional expressions are unique.} \)

**Example.** Write the PFD for 

\[
\frac{3x + 5}{(x - 13)(x + 14)}, \quad \text{according to the PFD with Linear Factors Theorem.}
\]

**Solution.** The degree of the numerator is 1, and the degree of the denominator is 2. Therefore, the degree of the numerator is less than the degree of the denominator. \((3x+5)\) and \((x+14)\) are non-repeating linear factors. Therefore, the partial fraction decomposition of the given fractional expression is:

\[
\frac{3x + 5}{(x - 13)(x + 14)} = \frac{A}{x - 13} + \frac{B}{x + 14},
\]

where \( A \) and \( B \) are constants that need to be determined. (We do not know how to find the constants yet.)

**Example.** Write the PFD for

\[
\frac{13x^3 - 2x + 4}{(x - 3)(x + 4)^3}, \quad \text{according to the PFD with Linear Factors Theorem.}
\]

**Solution.** The degree of the numerator is 3 and the degree of the denominator is 4. Therefore, the degree of the numerator is less than the degree of the denominator. \((x - 3)\) is a non-repeating linear factor and \((x + 4)\) is a repeating linear factor that repeats 3 times. Therefore, the partial fraction decomposition of the given fractional expression is:

\[
\frac{13x^3 - 2x + 4}{(x - 3)(x + 4)^3} = \frac{A_1}{x - 3} + \frac{A_2}{x + 4} + \frac{A_3}{(x + 4)^2} + \frac{A_4}{(x + 4)^3},
\]

where \( A_1, A_2, A_3, \) and \( A_4 \) are constants that need to be determined.
Let us say we want to find the constants in the previous two examples to complete the tasks.

**Example.** Find the PFD for

\[
\frac{3x + 5}{(x - 13)(x + 14)}
\]

according to the PFD with Linear Factors Theorem.

**Solution.**

\[
\frac{3x + 5}{(x - 13)(x + 14)} = \frac{A}{x - 13} + \frac{B}{x + 14}
\]

for some constants \( A \) and \( B \).

Right side

\[
\frac{A(x + 14) + B(x - 13)}{(x - 13)(x + 14)} = \frac{Ax + 14A + Bx - 13B}{(x - 13)(x + 14)} = \frac{(A + B)x + (14A - 13B)}{(x - 13)(x + 14)}
\]

According to the theorem, the fractional expression on the left side is equal to the fractional expression on the right side. Now, both rational expressions on the left side and on the right side have the same denominator. Therefore, numerators of those fractional expressions must be equal to each other. The only way the two numerators can be equal to each other is, if

\[
3 = A + B \quad (4.13)
\]

\[
5 = 14A - 13B
\]

The matrix form of the above system is:

\[
\begin{pmatrix}
1 & 1 & 3 \\
14 & -13 & 5
\end{pmatrix}
\]

By solving the system of equations, we can find \( A \) and \( B \). By using a (TI-83) calculator, \( A = \frac{44}{27} \) and \( B = \frac{37}{27} \).

Therefore, the PFD is:

\[
\frac{3x + 5}{(x - 13)(x + 14)} = \frac{44/27}{x - 13} + \frac{37/27}{x + 14}.
\]
CHAPTER 4. PARTIAL FRACTION DECOMPOSITION

or

\[
\frac{3x + 5}{(x - 13)(x + 14)} = \frac{44}{27(x - 13)} + \frac{37}{27(x + 14)}.
\]

Example. Find the PFD for

\[
\frac{13x^3 - 2x + 4}{(x - 3)(x + 4)^3}
\]

according to the PFD with Linear Factors Theorem.

Solution.

\[
\frac{13x^3 - 2x + 4}{(x - 3)(x + 4)^3} = \frac{A_1}{x - 3} + \frac{A_2}{x + 4} + \frac{A_3}{(x + 4)^2} + \frac{A_4}{(x + 4)^3},
\]

where, \(A_1, A_2, A_3,\) and \(A_4\) are constants.

Right side

\[
= \frac{A_1(x + 4)^3 + A_2(x - 3)(x + 4)^2 + A_3(x - 3)(x + 4) + A_4(x - 3)}{(x - 3)(x + 4)^3}
\]

\[
= \frac{(A_1 + A_2)x^3 + (12A_1 + 5A_2 + A_3)x^2 + (48A_1 - 8A_2 + A_3 + A_4)x + 64A_1 - 48A_2 - 12A_3 - 3A_4}{(x - 3)(x + 4)^3}
\]

By comparing fractional expressions, we get the following set of linear equations.

\[
\begin{align*}
13 &= A_1 + A_2 \\
0 &= 12A_1 + 5A_2 + A_3 \\
-2 &= 48A_1 - 8A_2 + A_3 + A_4 \\
4 &= 64A_1 - 48A_2 - 12A_3 - 3A_4
\end{align*}
\]

The matrix form of the above system is:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 13 \\
12 & 5 & 1 & 0 & 0 \\
48 & -8 & 1 & 1 & -2 \\
64 & -48 & -12 & -3 & 4
\end{pmatrix}
\]
4.2. PARTIAL FRACTION DECOMPOSITION: LINEAR FACTORS

By using a (TI-83) calculator, we get, $A_1 = \frac{349}{343}$, $A_2 = \frac{4110}{343}$, $A_3 = -\frac{3534}{49}$ and $A_4 = \frac{820}{7}$.

Therefore, the PFD is:

$$\frac{13x^3 - 2x + 4}{(x - 3)(x + 4)^3} = \frac{349}{343(x - 3)} + \frac{4110}{343(x + 4)} - \frac{3534}{49(x + 4)^2} + \frac{820}{7(x + 4)^3}.$$
4.3 Partial Fraction Decomposition: Irreducible Quadratic Factors

PFD with Irreducible Quadratic Factors Theorem. Suppose \( f(x) = \frac{p(x)}{q(x)} \) is a rational function, where \( p(x) \) and \( q(x) \) are polynomials and the degree of \( p(x) < \) the degree of \( q(x) \). Suppose \( q(x) \) can be completely factored into linear factors and irreducible quadratic factors.

1. For each linear factor, follow the PFD with Linear Factors Theorem.
2. For each non-repeating irreducible quadratic factor of \( q(x) \), write a fractional expression of the form \( \frac{A_k x + B_k}{\text{linear factor}} \), where \( A_k, B_k, \) etc., are constants.
3. For each repeating irreducible quadratic factor that repeats \( n \) times, write \( n \) factors of the form \( \frac{C_1 x + D_1}{\text{quadratic factor}}, \frac{C_2 x + D_2}{\text{quadratic factor}}^2, \ldots \frac{C_n x + D_n}{\text{quadratic factor}}^n \), where \( C_1, D_1, C_2, D_2, \ldots, C_n, D_n \) etc., are constants.

Then \( f(x) = \text{sum of all fractional expressions in (1), (2) and (3), and all those constants in the numerators of fractional expressions are unique.} \)

Example. Find the PFD of

\[
\frac{x^2 - 3x + 1}{(x - 1)(x^2 - x + 1)^2}.
\]

Solution. \( x^2 - x + 1 \) is an irreducible quadratic factor over \( \mathbb{R} \), because the discriminant is negative. Therefore, by the PFD with Quadratic Factors Theorem,

\[
\frac{x^2 - 3x + 1}{(x - 1)(x^2 - x + 1)^2} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 - x + 1} + \frac{Dx + E}{(x^2 - x + 1)^2},
\]

for some real constants \( A, B, C, D, \) and \( E \).

Right side

\[
= \frac{A(x^2 - x + 1)^2}{(x - 1)(x^2 - x + 1)^2} + \frac{(Bx + C)(x - 1)(x^2 - x + 1)}{(x - 1)(x^2 - x + 1)^2} + \frac{(Dx + E)(x - 1)}{(x - 1)(x^2 - x + 1)^2}
\]
4.3. PARTIAL FRACTION DECOMPOSITION: IRREDUCIBLE QUADRATIC FACTORS

\[
\frac{A(x^4 - 2x^3 + 3x^2 - 2x + 1) + (Bx + C)(x^3 - 2x^2 + 2x - 1) + (Dx + E)(x - 1)}{(x - 1)(x^2 - x + 1)^2}
\]

\[
\frac{Ax^4 - 2Ax^3 + 3Ax^2 - 2Ax + A + Bx^3 - 2Bx^2 + 2Bx^2 - Bx + Cx^3 - 2Cx^2 + 2Cx - C + Dx^2 - Dx + Ex - E}{(x - 1)(x^2 - x + 1)^2}
\]

\[
\frac{(A + B)x^4 + (-2A - 2B + C)x^3 + (3A + 2B - 2C + D)x^2 + (-2A - B + 2C - D + E)x + (A - C - E)}{(x - 1)(x^2 - x + 1)^2}
\]

By comparing fractional expressions, we get the following set of linear equations.

\[
\begin{align*}
0 &= A + B \\
0 &= -2A - 2B + C \\
1 &= 3A + 2B - 2C + D \\
-3 &= -2A - B + 2C - D + E \\
1 &= A - C - E
\end{align*}
\]

(4.17)

The matrix form of the above system is:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2 & -2 & 1 & 0 & 0 & 0 & 0 \\
3 & 2 & -2 & 1 & 0 & 1 & 1 \\
-2 & -1 & 2 & -1 & 1 & -3 & 1 \\
1 & 0 & -1 & 0 & -1 & 1 & 1
\end{pmatrix}
\]

(4.18)

By using a (TI-83) calculator we get, \(A = -1\), \(B = 1\), \(C = 0\), \(D = 2\) and \(E = -2\).

Therefore, the PFD is:

\[
\frac{x^2 - 3x + 1}{(x - 1)(x^2 - x + 1)^2} = \frac{-1}{x - 1} + \frac{x}{x^2 - x + 1} + \frac{2x - 2}{(x^2 - x + 1)^2}
\]
Exercise. Find the PFD of
\[
\frac{x^3 - 2x^2 - x + 1}{(x + 1)(x - 1)^2(x^2 + 1)^2}.
\]
Part II

Trigonometry
Chapter 5

Trigonometry of Angles

5.1 Angles

Consider two fixed rays with a common vertex $O$ on a plane. Now imagine another ray, we call it a free ray, with the same vertex, which is free to rotate around $O$. Suppose the free ray rotates from one fixed ray to the other fixed ray. The region generated by the sweeping free ray between the two fixed rays is called an angle. The point $O$ is called the vertex of the angle; the two fixed rays are called the sides of the angle: the fixed ray where the rotation began is called the initial side of the angle; and the fixed ray where the rotation ended is called the terminal side of the angle.

You should realize that there are infinitely many angles with the same vertex, the same initial side and the same terminal side. The following is another angle with the vertex $O$, the same initial side and the same terminal side as the previous angle.
When you rotate the free ray more than one full cycle from the initial ray to the terminal ray, then you get angles with overlapping regions. It is difficult to visualize overlapping regions. Therefore, from now on, we will indicate angles only by rotation.

A plane with the familiar \( xy \) coordinate system is known as the \( xy \)-plane. If we use a rigid transformation to move an angle so that its vertex coincides with the origin \( O \) and its initial side coincides with the positive \( x \)-axis, then we say that the angle is in the **standard position**. The following is another angle with the vertex \( O \), the same initial side and the same terminal side as the previous angle.

When you place angles in standard position, the angles with the same terminal side is known as **co-terminal angles**.

### 5.2 Degree Measure of an Angle

If we can somehow measure the “amount of rotation”, then we can distinguish all the different angles with the same vertex, the same initial side and the same terminal side. About 4000 years ago, the great civilizations of Middle East devised a method to measure angles. We use this measuring system developed by the Babylonians even today. The Babylonians had a base 60 number system. (It is possible that they used this system because it was easy to write fractions like \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \) and \( \frac{1}{6} \) in this system. For example, \( \frac{1}{2} = 30 \) units, \( \frac{1}{3} = 20 \) units, \( \frac{1}{4} = 15 \) units, \( \frac{1}{5} = 12 \) units and \( \frac{1}{6} = 10 \) units, in base
60 system\(^1\). Their ingenious method of measuring angles can be described as follows. Consider a circular disc. Partition the circumference of this into 360 equal parts by drawing tick marks. Then draw rays through each tick mark from $O$. The amount of rotation between any two consecutive rays is called 1 degree. By identifying tick marks with numbers $0, 1, 2, \ldots, 360$, now we have a tool to measure the rotation of an angle. To measure the rotation of an angle, place the circular disc on the same plane where the angle is so that the center of the disc coincides with $O$ and the 0 tick mark is on the initial side.

If the angle is less than a full counterclockwise rotation and if the terminal side lands on a tick mark (say, on 45\(^2\)) then we say the measure of the angle is 45 degrees. If the angle is 45 degrees more than one counterclockwise rotation, then we say the measure of

---

\(^1\)Read “Exact Sciences in antiquity” by Otto Neugebauer, if you are interested.

\(^2\)Most probably, the terminal side will not land on any tick mark. Then we use the tick mark closest to the terminal side.
the angle is $360 + 45 = 405$ degrees. In this manner, we can assign a degree measure to any angle with counterclockwise rotation. We consider clockwise rotations as negative rotations and assign a negative degree measure to those angles.

We usually use a little circle as a superscript to denote degrees. For example, $45^\circ$ stands for 45 degrees.

Notice that we have selected a circle with center $O$ when we defined the degree measure. There was no mention about the radius of the circle. That is, the chosen circle can have any radius as long as the center is at $O$. As you can imagine longer the radius, longer the circumference. Since we are partitioning the circle into exactly 360 equal parts, the longer the radius, the longer each part. Seemingly miraculously, for any given angle, the number of parts in an arc of a circle subtended by the angle is the same for any circle, no matter what the radius is.
You may have learned in high school that “all circles are similar”. That is the reason behind this seeming miracle. In that sense, the degree measure of an angle is unique. That is, there is only one degree measure for any given angle.

See also the Paper Folding Experiment.

5.3 Radian Measure of an Angle

There is another more modern angle measure known as the radian measure.

Consider an angle in standard position so that the rotation is counterclockwise. Once again, place a circular disk with radius $r$ (in distance units) so that its center is $O$. Suppose the arc-length of the arc subtended by the angle is $s$ in distance units. The ratio $\frac{s}{r}$ is called the radian measure of the angle. The radian measure of an angle is also unique since all circles are similar.

The circle with center $(0,0)$ and radius 1 unit on the $xy$-plane is called the unit circle.
Since the radius of the circle we pick is irrelevant when we measure an angle in the standard position (as long as the center of the circle is O), we can use the unit circle to measure angles.

If the arc on the unit circle subtended by an angle in a standard position has an arc-length of 1 unit, then the radian measure of the angle is 1 radian.

5.4 Relationship between the Degree Measure and the Radian Measure

You may have learned in middle school or high school that the circumference of a circle with radius $r$ is $2\pi r$. Accordingly, the circumference of a unit circle is $2\pi$. By the definition of the radian measure, the radian measure of a full angle is $2\pi$. When the circumference of the unit circle is partitioned into 360 equal parts, then the length of one part is $\frac{2\pi}{360}$ or $\frac{\pi}{180}$. Therefore, the radian measure of an angle of $1^\circ$ is $\frac{\pi}{180}$; the radian measure of an angle of $2^\circ$ is $\frac{2\pi}{180}$; the radian measure of an angle of $3^\circ$ is $\frac{3\pi}{180}$; the radian measure of an angle of $-3^\circ$ is $\frac{-3\pi}{180}$; etc; and the radian measure of an angle of $n^\circ$ is $\frac{n\pi}{180}$, where $n$ is an integer.

It is a convention to use Greek lowercase letters to represent angles. For example, $\alpha$ (alpha), $\beta$ (beta), $\gamma$ (gamma), $\theta$, (theta) are a few such letters that are often used to represent angles.

Consider the following angles. Let $\theta_1$ be the angle in the standard position, where the initial and the terminal sides coincide. Suppose we partition the circumference of the unit circle into 12 equal parts and one of the partition tick-marks coincides with the positive $x$-axis. Let $\theta_2$ be the angle in the standard position, where the terminal side passes through the first tick-mark in the counterclockwise direction. Suppose we partition the circumference of the unit circle into 8 equal parts and one of the partition tick-marks coincides with the positive $x$-axis. Let $\theta_3$ be the angle in the standard position, where the terminal side passes through the first tick-mark in the counterclockwise direction. Suppose we partition the circumference of the unit circle into 6 equal parts and one of the partition tick-marks coincides with the positive $x$-axis. Let $\theta_4$ be the angle in the standard position, where the terminal side passes through the first tick-mark in the counterclockwise direction. Suppose we partition the circumference of the unit circle into 4 equal parts and one of the partition tick-marks coincides with the positive $x$-axis. Let $\theta_5$ be the angle in the standard position, where the terminal side pass through the first
5.5. TRIGONOMETRIC NUMBERS OF ANGLES

The following table lists the angle measure and the radian measure of the chosen angles.
(These angles will be known as “special angles”, for lack of a better term, from now on.)

<table>
<thead>
<tr>
<th>Angle</th>
<th>Degree Measure</th>
<th>Radian Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>30</td>
<td>$\frac{\pi}{6}$</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>45</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>60</td>
<td>$\frac{\pi}{3}$</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>90</td>
<td>$\frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

5.5 Trigonometric Numbers of Angles

We will define two important numbers for a given angle $\theta$ in the standard position as follows. Let $P(a, b)$ be any chosen point on the terminal side of $\theta$ other than $O$. Let $r$ be the length of $OP$. We define $\sin \theta$, pronounced “sine theta”, as the ratio $\frac{b}{r}$ and we define $\cos \theta$, pronounced “cosine theta”, as the ratio $\frac{a}{r}$. The $\sin \theta$ and the $\cos \theta$ are two of the six trigonometric numbers of $\theta$ that we are going to define for a given angle $\theta$.

Notice that in the definition of $\sin \theta$, we are free to choose the point $P$ on the terminal side of the angle $\theta$ as long as $P \neq O$. Can the value of the number $\sin \theta$ be different for two different points $P_1(a_1, b_1)$ and $P_2(a_2, b_2)$ on the terminal side of the angle $\theta$? We will answer this question if the terminal side of $\theta$ lies in the third quadrant. Finding answers to other quadrants and the axes will be left as an exercise.

Suppose the terminal side of $\theta$ is in the third quadrant. Then both $b_1$, $b_2$ are negative. Therefore, the sign of $\frac{b_1}{r_1}$ is the same as the sign of $\frac{b_2}{r_2}$. 

tick-mark in the counterclockwise direction.
Now drop perpendiculars from both $P_1$ and $P_2$ onto the $x$-axis. Let the feet of those perpendiculars be $Q_1$ and $Q_2$ respectively.

![Diagram showing perpendiculars from $P_1$ and $P_2$ to the $x$-axis]

The triangles $OP_1Q_1$ and $OP_2Q_2$ are similar by the AA criteria. Therefore, $\frac{P_1Q_1}{P_1O} = \frac{P_2Q_2}{P_2O}$. That is, $\frac{-b_1}{r_1} = \frac{-b_2}{r_1}$. Therefore, the number $\sin \theta$ is the same for both $P_1$ and $P_2$. Since $P_1$ and $P_2$ are two arbitrary points on the terminal side of $\theta$, this observation implies that, by definition, the number $\sin \theta$ is unique for any point selected on the terminal side of $\theta$.

Exercise.

1. Show that the number $\sin \theta$ is unique if $\theta$ is in the first quadrant.
2. Show that the number $\sin \theta$ is unique if $\theta$ is in the second quadrant.
3. Show that the number $\sin \theta$ is unique if $\theta$ is in the fourth quadrant.
4. Show that the number $\sin \theta$ is unique if $\theta$ is on the positive $x$-axis.
5. Show that the number $\sin \theta$ is unique if $\theta$ is on the negative $x$-axis.
6. Show that the number $\sin \theta$ is unique if $\theta$ is on the positive $y$-axis.
7. Show that the number $\sin \theta$ is unique if $\theta$ is on the negative $y$-axis.
You can show by similar means that the number $\cos \theta$ is also unique for a given angle $\theta$, by definition.

It is customary to replace the name of the angle by its measure if we know the measure. For example, we usually write $\sin \frac{\pi}{3}$ for $\sin \theta$, when $\theta$ has the measure $\frac{\pi}{3}$. We will also loosely say $\theta = \frac{\pi}{3}$.

Some of the theorems that you may have learned in a high school geometry class are going to be very useful. The following is a list of high school geometric theorems. We will use the following notation in these theorems. Consider two points $A$ and $B$. We will use $AB$ to represent the line segment with endpoints $A$ and $B$. We will use $|AB|$ for the length of $AB$. We will use $L_{AB}$ for the line containing $A$ and $B$.

**Angle Sum of a Triangle Theorem.** The sum of the measures of the angles of a triangle is $180^\circ$.

**Equilateral Triangle Theorem.** The measure of any angle of an equilateral triangle is $60^\circ$, and vice versa.

**Isosceles Triangle Theorem.** The measures of angles facing congruent sides of an isosceles triangle are equal, and vice versa.

**Perpendicular Bisector Theorem.** Let $\triangle ABC$ be an isosceles triangle so that $AB \cong AC$. Then the angle bisector of $\angle A$ is the perpendicular bisector of $BC$, and vice versa.

**SAS Theorem.** Let $\triangle ABC$ and $\triangle DEF$ are two triangles so that $AB \cong DE$, $\angle B \cong \angle E$ and $BC \cong EF$. Then the two triangles are congruent.
CHAPTER 5. TRIGONOMETRY OF ANGLES

Pythagorean Theorem. Let $\triangle ABC$ be a right triangle so that $AC$ is the hypotenuse. Then $|AC|^2 = |AB|^2 + |BC|^2$.

Now we will derive the sine and cosine numbers of “special” angles. We will use the radian measure for an angle in the following theorems.

**Trigonometric Numbers of 0 Theorem.**

1. $\sin 0 = 0$
2. $\cos 0 = 1$

**Proof.** Let $\theta$ be the given angle.

Since the measure of $\theta = 0$, the terminal side of $\theta$ lies on the positive $x$-axis. Let $P(a,0)$ be any arbitrary point on the terminal side of $\theta$, where $a \neq 0$. Then $r = |OP| = a$, and, by definition, $\sin 0 = \frac{0}{a} = 0$. Also, $\cos 0 = \frac{a}{r}$, by definition. Since $a \neq 0$, $\frac{a}{r} = \frac{a}{a} = 1$. Therefore, $\cos 0 = 1$.\qed
Trigonometric Numbers of $\frac{\pi}{6}$ Theorem.

1. $\sin \frac{\pi}{6} = \frac{1}{2}$
2. $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Proof. Let $\theta$ be the angle with measure $\frac{\pi}{6}$. Let $P(a, b)$ be an arbitrary point on the terminal side of $\theta$ so that $a \neq 0$ and $b \neq 0$, and let $|OP| = r$.

Drop a perpendicular from $P$ to the positive $x$-axis. Let the foot of this perpendicular be $R$. The angle $\angleROP$ has the measure $\frac{\pi}{6}$ because it is $\theta$. Then the measure of the angle $\angleOPR$ is $\frac{\pi}{3}$, since the angle $ORP$ is a right angle and by the Angle Sum of a Triangle Theorem.

Now extend $PR$ to $Q$ so that $|PR| = |RQ|$. Connect $O$ and $Q$ by a line segment.
By the SAS theorem, the triangle $\triangle OPR$ and the triangle $\triangle OQR$ are congruent. ($PR \cong QR$, by construction; $OR$ is a common side; and angles $\angle ORP$ and $\angle ORQ$ are right angles.) Therefore, $\angle OQR$ has measure $\frac{\pi}{3}$ and the triangle $\triangle OPQ$ is an equilateral triangle. Then $|PQ| = r$, since $|OP| = r$. Also, $L_{OR}$ is the perpendicular bisector of $PQ$. Therefore, $|PR| = \frac{1}{2}r$. By the Pythagorean Theorem, $|OR|^2 = |OP|^2 - |PR|^2$. Therefore, $|OR|^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$. That is, $|OR| = \frac{\sqrt{3}}{2}r$. Therefore, $a = \frac{\sqrt{3}}{2}r$ and $b = \frac{1}{2}r$. By definition, $\sin \theta = \frac{b}{r} = \frac{1}{2}$ and $\cos \theta = \frac{a}{r} = \frac{\sqrt{3}}{2}$.

### Trigonometric Numbers of $\frac{\pi}{4}$ Theorem.

1. $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$
2. $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

**Proof.** Let $\theta$ be the angle with measure $\frac{\pi}{4}$. Let $P(a, b)$ be a point on the terminal side of $\theta$ so that both $a$ and $b$ are positive and let $|OP| = r$. 

---

By the SAS theorem, the triangle $\triangle OPR$ and the triangle $\triangle OQR$ are congruent. ($PR \cong QR$, by construction; $OR$ is a common side; and angles $\angle ORP$ and $\angle ORQ$ are right angles.) Therefore, $\angle OQR$ has measure $\frac{\pi}{3}$ and the triangle $\triangle OPQ$ is an equilateral triangle. Then $|PQ| = r$, since $|OP| = r$. Also, $L_{OR}$ is the perpendicular bisector of $PQ$. Therefore, $|PR| = \frac{1}{2}r$. By the Pythagorean Theorem, $|OR|^2 = |OP|^2 - |PR|^2$. Therefore, $|OR|^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$. That is, $|OR| = \frac{\sqrt{3}}{2}r$. Therefore, $a = \frac{\sqrt{3}}{2}r$ and $b = \frac{1}{2}r$. By definition, $\sin \theta = \frac{b}{r} = \frac{1}{2}$ and $\cos \theta = \frac{a}{r} = \frac{\sqrt{3}}{2}$.

### Trigonometric Numbers of $\frac{\pi}{4}$ Theorem.

1. $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$
2. $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

**Proof.** Let $\theta$ be the angle with measure $\frac{\pi}{4}$. Let $P(a, b)$ be a point on the terminal side of $\theta$ so that both $a$ and $b$ are positive and let $|OP| = r$. 

---

By the SAS theorem, the triangle $\triangle OPR$ and the triangle $\triangle OQR$ are congruent. ($PR \cong QR$, by construction; $OR$ is a common side; and angles $\angle ORP$ and $\angle ORQ$ are right angles.) Therefore, $\angle OQR$ has measure $\frac{\pi}{3}$ and the triangle $\triangle OPQ$ is an equilateral triangle. Then $|PQ| = r$, since $|OP| = r$. Also, $L_{OR}$ is the perpendicular bisector of $PQ$. Therefore, $|PR| = \frac{1}{2}r$. By the Pythagorean Theorem, $|OR|^2 = |OP|^2 - |PR|^2$. Therefore, $|OR|^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$. That is, $|OR| = \frac{\sqrt{3}}{2}r$. Therefore, $a = \frac{\sqrt{3}}{2}r$ and $b = \frac{1}{2}r$. By definition, $\sin \theta = \frac{b}{r} = \frac{1}{2}$ and $\cos \theta = \frac{a}{r} = \frac{\sqrt{3}}{2}$.
Drop a perpendicular from $P$ to the $x$-axis so that $R$ is the foot of the perpendicular. The angle $\angle OPR$ has the same measure as $\theta$ as a result of the Angle Sum of a Triangle Theorem.

Then $\triangle OPR$ is an isosceles triangle with $OR \cong PR$. This means $a = b$, since the coordinates of $P$ are $(a, b)$. By the Pythagorean Theorem, $r^2 = a^2 + a^2$. Therefore, $a = b = \frac{1}{\sqrt{2}}r$. By definition, $\sin \theta = \frac{b}{r} = \frac{1}{\sqrt{2}}$ and $\cos \theta = \frac{a}{r} = \frac{1}{\sqrt{2}}$. \[\square\]

**Trigonometric Numbers of $\frac{\pi}{3}$ Theorem.**

1. $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
2. $\cos \frac{\pi}{3} = \frac{1}{2}$

**Proof.** Let $\theta$ be the angle with measure $\frac{\pi}{3}$. Let $P(a, b)$ be a point on the terminal side of $\theta$ so that both $a$ and $b$ are positive and let $|OP| = r$. 
Drop a perpendicular from $P$ to the $x$-axis so that $R$ is the foot of the perpendicular. Extend $OR$ to $Q$ so that $|OR| = |RQ|$. Connect $P$ and $Q$ by a line segment.

The line $L_{PR}$ is the perpendicular bisector of $OQ$. Therefore, $L_{PR}$ is the angle bisector of $\angle OPQ$. The angle $\angle OPR$ has the measure $\frac{\pi}{6}$ by the Angle Sum of a Triangle Theorem on the triangle $\triangle OPR$. Then, $\angle RPQ$ has the measure $\frac{\pi}{6}$ and therefore, $\angle OPQ$ has the measure $\frac{\pi}{3}$. Then by the Angle Sum of a Triangle Theorem, $\angle PQO$ has the measure $\frac{\pi}{3}$ and the triangle $POQ$ is an equilateral triangle, by the Equilateral Triangle Theorem. Then $|OQ| = r$ and $|OR| = \frac{1}{2}r$. By the Pythagorean Theorem, $|PR|^2 = |OP|^2 - |OR|^2$. That is, $|PR|^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$. Therefore, $|PR| = \frac{\sqrt{3}}{2}$. By definition, $\sin \theta = \frac{b}{r} = \frac{\sqrt{3}}{2}$ and $\cos \theta = \frac{a}{r} = \frac{1}{2}$. 

\[\square\]
Trigonometric Numbers of $\frac{\pi}{2}$ Theorem.

1. $\sin \frac{\pi}{2} = 1$
2. $\cos \frac{\pi}{2} = 0$

Proof. Let $\theta$ be the given angle. Let $P(0, b)$ be a point on the terminal side of $\theta$, where $b \neq 0$. Let $|OP| = r$. Then $r = b$.

By definition, $\sin \frac{\pi}{2} = \frac{b}{r} = \frac{b}{b} = 1$ since $b \neq 0$. Also, $\cos \frac{\pi}{2} = \frac{a}{r} = \frac{0}{b} = 0$, by definition.

The results of the last five theorems are summarized in the following table. That is, the sine and cosine trigonometric numbers of the special angles are:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Here is another theorem that you may have learned in high school.
**The Two Dimensional Distance Formula Theorem.** Let $P(x_1, y_1)$ and $Q(x_1, y_1)$ be two points. Then the distance between $P$ and $Q$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

(You can prove the Distance Formula Theorem using the Pythagorean Theorem.)

**Example.** Suppose an angle $\theta$ is in the standard position and $P(-2, 3)$ is a point on the terminal side of $\theta$. Find $\sin \theta$ and $\cos \theta$.

**Answer.** Let $|OP| = r$. Then by the Distance Formula Theorem, $r = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{13}$. Therefore, $\sin \theta = \frac{3}{\sqrt{13}}$ and $\cos \theta = \frac{-2}{\sqrt{13}}$.

In the previous example, the terminal side of the given angle $\theta$ lies in the second quadrant. It turned out that $\sin \theta > 0$ and $\cos \theta < 0$. This is in fact, a true statement for any angle in standard position if the terminal side lies in the second quadrant. The following is a theorem that you can prove using just the definitions of $\sin \theta$ and $\cos \theta$.

We will loosely say “the angle is in the second quadrant” if the angle is in the standard position and its terminal side lies in the second quadrant.

**Theorem.** Let $\theta$ be an angle in standard position.

1. If $\theta$ is in the first quadrant, then $\sin \theta > 0$ and $\cos \theta > 0$.
2. If $\theta$ is in the second quadrant, then $\sin \theta > 0$ and $\cos \theta < 0$.
3. If $\theta$ is in the third quadrant, then $\sin \theta < 0$ and $\cos \theta < 0$.
4. If $\theta$ is in the fourth quadrant, then $\sin \theta < 0$ and $\cos \theta > 0$.

By the definitions of $\sin \theta$ and $\cos \theta$, for a given angle $\theta$ in a standard position, it does not matter what point $P$ we select on the terminal side to compute those numbers. We could choose $P$ as the point where the terminal side of $\theta$ intersects the unit circle. Since the radius of the unit cycle is 1, $r = 1$. Therefore, for a given angle $\theta$ in standard position, if we choose the point of intersection of the terminal side and the unit circle as $P(a, b)$, then $\sin \theta = \frac{b}{r} = b$, $\cos \theta = \frac{a}{r} = a$. That is, the coordinates of $P$ are $(\cos \theta, \sin \theta)$.
In the following figure, the measure of the angle is written on the terminal side of the angle, and the coordinates of the point on the unit circle is shown. We have used the trigonometric numbers of the special angles in this figure.

With this figure in place, we can find trigonometric numbers of more angles related to the special angles.

The reflection of the terminal side of the angle $\frac{\pi}{4}$ with respect to $y$-axis is the terminal side of the angle $\frac{3\pi}{4}$. Therefore, $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$.

The reflection of the terminal side of the angle $\frac{\pi}{4}$ with respect to $x$-axis is the terminal side of the angle $\frac{7\pi}{4}$. Therefore, $\sin \frac{7\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$.

The reflection of the terminal side of the angle $\frac{3\pi}{4}$ with respect to $x$-axis is the terminal side of the angle $\frac{5\pi}{4}$. Therefore, $\sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\cos \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$. 
These results are given in the following figure.

The reflection of the terminal side of the angle $\frac{\pi}{6}$ with respect to $y$-axis is the terminal side of the angle $\frac{5\pi}{6}$. Therefore, $\sin \frac{5\pi}{6} = \frac{1}{2}$ and $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$.

The reflection of the terminal side of the angle $\frac{\pi}{6}$ with respect to $x$-axis is the terminal side of the angle $\frac{11\pi}{6}$. Therefore, $\sin \frac{11\pi}{6} = -\frac{1}{2}$ and $\cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2}$.

The reflection of the terminal side of the angle $\frac{5\pi}{6}$ with respect to $x$-axis is the terminal side of the angle $\frac{7\pi}{6}$. Therefore, $\sin \frac{7\pi}{6} = -\frac{1}{2}$ and $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$.

These results are given in the following figure.

The reflection of the terminal side of the angle $\frac{\pi}{3}$ with respect to $y$-axis is the terminal
side of the angle \( \frac{2\pi}{3} \). Therefore, \( \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \) and \( \cos \frac{2\pi}{3} = -\frac{1}{2} \).

The reflection of the terminal side of the angle \( \frac{\pi}{3} \) with respect to \( x \)-axis is the terminal side of the angle \( \frac{5\pi}{3} \). Therefore, \( \sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2} \) and \( \cos \frac{5\pi}{3} = \frac{1}{2} \).

The reflection of the terminal side of the angle \( \frac{2\pi}{3} \) with respect to \( x \)-axis is the terminal side of the angle \( \frac{4\pi}{3} \). Therefore, \( \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \) and \( \cos \frac{4\pi}{3} = -\frac{1}{2} \).

These results are given in the following figure.

The following is a figure that contains all of our findings and the trigonometric values of angles in the standard position.
Chapter 6

Trigonometric Functions

We will use the radian measure for angles in this section. We will also use \( \theta \) to represent the radian measure of the angle \( \theta \). Since the radian measure of a given angle is a unique number, \( \theta \) is a number. By the definition of \( \sin \theta \), there is only one number \( \sin \theta \), for a given angle \( \theta \). Let \( f(\theta) = \sin \theta \). Then \( f(\theta) \) is a function of \( \theta \), because for each angle \( \theta \) there is exactly one \( \sin \theta \).

Consider a \( \theta y \) coordinate system (In this system, \( \theta \)-axis plays the role usually played by the \( x \)-axis. As a matter of fact, we could replace \( \theta \) with \( x \) and use the usual \( xy \) coordinate system. But we will wait until we are a little more comfortable with sine and cosine functions before making this move.)

6.1 Sine Function

The collection of all points \((\theta, \sin \theta)\) on the \( \theta y \)-plane is called the graph of the function \( f(\theta) = \sin \theta \).

One observation about the graph of \( f(\theta) = \sin \theta \) is extremely useful. Since \( \sin \theta \) is the \( y \)-coordinate of the intersection of the terminal side and the unit circle, the values of the sine function are the same for co-terminal angles. Co-terminal angles differ from each other by an integer multiple of a full cycle. That is, if \( \theta \) and \( \alpha \) are co-terminal angles, then there is an integer \( k \) so that \( \alpha = 2k\pi + \theta \). Therefore,

\[
\sin(2k\pi + \theta) = \sin \theta \text{ for any integer } k.
\]

That is, the graph of \( f(\theta) = \sin \theta \) in the interval \([2k_1\pi, 2(k_1 + 1)\pi]\), for a given integer \( k_1 \), is identical to the graph of \( f(\theta) = \sin \theta \) in the interval \([2k\pi, 2(k + 1)\pi]\), for any integer \( k \).
Therefore, if we can sketch the graph of \( f(\theta) = \sin \theta \) in the interval \([0, 2\pi]\), then we know the graph of \( f(\theta) = \sin \theta \) for the entire \( \theta \)-axis.

We will use the sine numbers of special angles and the reflections of special angles to get a rough sketch of the graph of \( f(\theta) = \sin \theta \) in the interval \([0, 2\pi]\). The following table contains sine numbers of special angles and the reflections of special angles.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \theta )</th>
<th>( \sin \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>( \pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{6} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{7\pi}{6} )</td>
<td>-( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{3\pi}{4} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{5\pi}{4} )</td>
<td>-( \frac{1}{\sqrt{2}} )</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{4\pi}{3} )</td>
<td>-( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{11\pi}{6} )</td>
<td>-( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{3\pi}{2} )</td>
<td>( 2\pi )</td>
<td>0</td>
</tr>
</tbody>
</table>

The sketch of the graph of \( f(\theta) = \sin \theta \) in the interval \([0, 2\pi]\) is given below.

The shape of the graph of \( f(\theta) = \sin \theta \) over \([0, 2\pi]\) is congruent to the shape of the graph of \( f(\theta) = \sin \theta \) over \([2k\pi, 2(k+1)\pi]\), for any integer \( k \), by the definition of \( \sin \theta \) on the unit circle.

The graph of \( f(\theta) = \sin \theta \) contains repeating patterns. Clearly, the portion of the graph of \( f(\theta) = \sin \theta \) over \([0, 2\pi]\) is a repeating pattern.
6.1. SINE FUNCTION

The portion of the graph of \( f(\theta) = \sin \theta \) over \([-\pi/2, 3\pi/2]\) is also a repeating pattern.

The portion of the graph of \( f(\theta) = \sin \theta \) over \([-2\pi, 2\pi]\) is also a repeating pattern.

It is not difficult to see that there are infinitely many repeating patterns on the graph of \( f(\theta) = \sin \theta \). If the graph of a function contains repeating patterns, then we say the function is periodic. The length of the smallest interval over a pattern of the graph resides is called the period of a periodic function. That is, if \( p \) is the period of a periodic function \( f \), then \( p \) is the smallest value with the property: for any number \( \alpha \), \( f(p + \alpha) = f(\alpha) \). We already know that \( \sin(2k\pi + \theta) = \sin \theta \), for any integer \( k \) and for any number \( \theta \).
Exercise.

1. Prove that the period of \( f(\theta) = \sin \theta \) is \( 2\pi \).
2. Prove that the maximum value of \( f(\theta) = \sin \theta \) is 1.
3. Prove that the minimum value of \( f(\theta) = \sin \theta \) is \(-1\).

Suppose a periodic function has a maximum value and a minimum value. One half of (the maximum value – the minimum value) of a periodic function is called the amplitude of the periodic function.

Exercise.

1. Prove that the amplitude of \( f(\theta) = \sin \theta \) is 1.
2. Prove that the domain of \( f(\theta) = \sin \theta \) is \((-\infty, \infty)\) and the range is \([-1, 1]\).

It should be clear to you that if we can sketch the portion of the graph of a periodic function over an interval with a length equal to the period, then we can get the rest of the graph of the function. From now on, we will concentrate our attention on only a portion of the graph a periodic function over an interval whose length is the period of the periodic function. For the sine function, we will select the interval \([0, 2\pi]\). We will loosely say that we are sketching the graph of the function over a period.

The following are a few important observations of the above graph.

1. The sine function has zeros at 0, \( \pi \), and \( 2\pi \) in \([0, 2\pi]\).
2. The sine function has the maximum value at \( \frac{\pi}{2} \) in \([0, 2\pi]\).
3. The sine function has the minimum value at \( \frac{3\pi}{2} \) in \([0, 2\pi]\).

We say a function \( f \) is symmetrical with respect to the vertical line \( x = \alpha \) in an interval \([a, b]\) if \( f(\theta - \alpha) = f(\theta + \alpha) \) in \([a, b]\).
Exercise.

1. Prove that \( f(\theta) = \sin \theta \) is symmetrical with respect to \( x = \frac{\pi}{2} \) in the interval \([0, \pi]\).

2. Prove that \( f(\theta) = \sin \theta \) is symmetrical with respect to \( x = \frac{3\pi}{2} \) in the interval \([\pi, 2\pi]\).

Example. Identify the amplitude, the period, and sketch the graph of \( f_1(\theta) = 3\sin \theta \) over one period.

Answer. The function value of \( f_1 \) at any given \( \theta \) is three times the value of \( \sin \theta \) for any \( \theta \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( 3 \sin \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
The amplitude of $f_1$ is 3 and the period is $2\pi$.

**Example.** Identify the amplitude, the period, and sketch the graph of $f_2(\theta) = \sin 3\theta$ over one period.

**Answer.** Let $\alpha = 3\theta$. Then $\theta = \frac{\alpha}{3}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sin \alpha$</th>
<th>$\theta = \frac{\alpha}{3}$</th>
<th>$\sin 3\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>1</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>$\frac{\pi}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>$-1$</td>
<td>$\frac{\pi}{2}$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
<td>$\frac{2\pi}{3}$</td>
<td>0</td>
</tr>
</tbody>
</table>
The amplitude of $f_2$ is 1 and the period is $\frac{2\pi}{3}$.

**Example.** Identify the amplitude, the period, and sketch the graph of $f_3(\theta) = 3\sin 3\theta$ over one period.

**Answer.** Let $\alpha = 3\theta$. Then $\theta = \frac{\alpha}{3}$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sin \alpha$</th>
<th>$\theta = \frac{\alpha}{3}$</th>
<th>$3\sin 3\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td>$\frac{\pi}{6}$</td>
<td>3</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>$\frac{\pi}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>-1</td>
<td>$\frac{\pi}{2}$</td>
<td>-3</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
<td>$\frac{2\pi}{3}$</td>
<td>0</td>
</tr>
</tbody>
</table>

![Graph](image-url)
The amplitude of \( f_3 \) is 3 and the period is \( \frac{2\pi}{3} \).

**Example.** Identify the amplitude, the period, and sketch the graph of \( f_4(\theta) = \sin(\theta - \frac{\pi}{3}) \) over one period.

**Answer.** Let \( \alpha = \theta - \frac{\pi}{3} \). Then \( \theta = \alpha + \frac{\pi}{3} \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \sin \alpha )</th>
<th>( \theta = \alpha + \frac{\pi}{3} )</th>
<th>( \sin(\theta - \frac{\pi}{3}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \frac{\pi}{3} )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>1</td>
<td>( \frac{5\pi}{6} )</td>
<td>1</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>( \frac{4\pi}{3} )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>-1</td>
<td>( \frac{11\pi}{6} )</td>
<td>-1</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>0</td>
<td>( \frac{7\pi}{3} )</td>
<td>0</td>
</tr>
</tbody>
</table>

The amplitude of \( f_4 \) is 1 and the period is \( 2\pi \).

Notice that if the graph of the sine function is shifted to the right by \( \frac{\pi}{3} \), then we get the graph of \( f_4(\theta) = \sin(\theta - \frac{\pi}{3}) \). We say \( +\frac{\pi}{3} \) is the phase shift of \( f_4 \).

**Example.** Identify the amplitude, the period, and sketch the graph of \( f_5(\theta) = \sin(\theta + \frac{\pi}{3}) \) over one period.

**Answer.** Let \( \alpha = \theta + \frac{\pi}{3} \). Then \( \theta = \alpha - \frac{\pi}{3} \).
6.1. SINE FUNCTION

\[ \alpha \sin \alpha \theta = \alpha - \frac{\pi}{3} \sin(\theta + \frac{\pi}{3}) \]

\[ \begin{array}{|c|c|c|c|}
\hline
\alpha & \sin \alpha & \theta = \alpha - \frac{\pi}{3} & \sin(\theta + \frac{\pi}{3}) \\
\hline
0 & 0 & -\frac{\pi}{3} & 0 \\
\frac{\pi}{2} & 1 & \frac{\pi}{6} & 1 \\
\pi & 0 & \frac{2\pi}{3} & 0 \\
\frac{3\pi}{2} & -1 & \frac{7\pi}{6} & -1 \\
2\pi & 0 & \frac{5\pi}{3} & 0 \\
\hline
\end{array} \]

The amplitude of \( f_5 \) is 1 and the period is \( 2\pi \).

Notice that if the graph of the sine function is shifted to the left by \( \frac{\pi}{3} \) (or to the right by \( -\frac{\pi}{3} \)), then we get the graph of \( f_5(\theta) = \sin(\theta + \frac{\pi}{3}) \). We say the phase shift of \( f_5 \) is \( -\frac{\pi}{3} \).

**Example.** Identify the amplitude, the period, the phase shift, and sketch the graph of \( f_6(\theta) = 3\sin 3(\theta + \frac{\pi}{3}) \) over one period.

**Answer.** Let \( \alpha = 3(\theta + \frac{\pi}{3}) \). Then \( \theta = \frac{\alpha}{3} - \frac{\pi}{3} \).

\[ \begin{array}{|c|c|c|c|c|}
\hline
\alpha & \sin \alpha & \frac{\alpha}{3} & \theta = \frac{\alpha}{3} - \frac{\pi}{3} & \sin(\theta + \frac{\pi}{3}) & 3\sin(\theta + \frac{\pi}{3}) \\
\hline
0 & 0 & 0 & -\frac{\pi}{3} & 0 & 0 \\
\frac{\pi}{2} & 1 & \frac{\pi}{6} & -\frac{\pi}{6} & 1 & 3 \\
\pi & 0 & \frac{\pi}{3} & 0 & 0 & 0 \\
\frac{3\pi}{2} & -1 & \frac{\pi}{2} & \frac{\pi}{6} & -1 & -3 \\
2\pi & 0 & \frac{2\pi}{3} & \frac{\pi}{3} & 0 & 0 \\
\hline
\end{array} \]
The amplitude of \( f_6 \) is 3, the period is \( \frac{2\pi}{3} \), and the phase shift is \( -\frac{\pi}{3} \).

### 6.2 Cosine Function

The collection of all points \((\theta, \cos \theta)\) on the \( \theta y \)-plane is called the graph of the function \( f(\theta) = \cos \theta \).

Since \( \cos \theta \) is the \( x \)-coordinate of the intersection of the terminal side and the unit circle, the values of the cosine function are the same for co-terminal angles. Each co-terminal angle differs from each other by an integer multiple of a full cycle. That is, if \( \theta \) and \( \alpha \) are co-terminal angles, then there is an integer \( k \) so that \( \alpha = 2k\pi + \theta \). Therefore,

\[
\cos(2k\pi + \theta) = \cos \theta \text{ for any integer } k.
\]

That is, the graph of \( f(\theta) = \cos \theta \) in the interval \([2k_1\pi, 2(k_1 + 1)\pi]\), for a given integer \( k_1 \), is identical to the graph of \( f(\theta) = \cos \theta \) in the interval \([2k\pi, 2(k + 1)\pi]\), for any integer \( k \).
6.2. COSINE FUNCTION

Therefore, if we can sketch the graph of \( f(\theta) = \cos \theta \) in the interval \([0, 2\pi]\), then we know the graph of \( f(\theta) = \cos \theta \) for the entire \( \theta \)-axis.

We will use the cosine numbers of special angles and the reflections of special angles to get a rough sketch of the graph of \( f(\theta) = \cos \theta \) in the interval \([0, 2\pi]\). The following table contains the cosine numbers of special angles and the reflections of special angles.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \cos \theta )</th>
<th>( \theta )</th>
<th>( \cos \theta )</th>
<th>( \theta )</th>
<th>( \cos \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
<td>( \pi )</td>
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</tr>
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<tr>
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<td>( \frac{3\pi}{4} )</td>
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<td>( \frac{5\pi}{4} )</td>
<td>-( \frac{1}{\sqrt{2}} )</td>
</tr>
<tr>
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<td>( \frac{1}{2} )</td>
<td>( \frac{5\pi}{6} )</td>
<td>-( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{4\pi}{3} )</td>
<td>-( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \cos \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3\pi}{2} )</td>
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<tr>
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<td>( \frac{1}{2} )</td>
</tr>
<tr>
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<td>( \frac{1}{\sqrt{2}} )</td>
</tr>
<tr>
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<td>( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>1</td>
</tr>
</tbody>
</table>

The sketch of the graph of \( f(\theta) = \cos \theta \) in the interval \([0, 2\pi]\) is given below.

The cosine function is also a periodic function with period \( 2\pi \). The amplitude of the cosine function is 1, the maximum value of the cosine function is 1, and the minimum value of the cosine function is \(-1\).
CHAPTER 6. TRIGONOMETRIC FUNCTIONS

Exercise.

1. Prove that the period of \( f(\theta) = \cos \theta \) is \( 2\pi \).
2. Prove that the maximum value of \( f(\theta) = \cos \theta \) is 1.
3. Prove that the minimum value of \( f(\theta) = \cos \theta \) is \(-1\).
4. Prove that the amplitude of \( f(\theta) = \cos \theta \) is 1.
5. Prove that the domain of \( f(\theta) = \cos \theta \) is \(( -\infty, \infty )\) and the range is \([-1, 1]\).

The graph of the cosine function in the interval \([0, 2\pi]\) is given below.

![Graph of the cosine function in the interval \([0, 2\pi]\)]

The following are a few important observation of the above graph.

1. The cosine function has zeros at \( \frac{\pi}{2} \) and at \( \frac{3\pi}{2} \) in \([0, 2\pi]\).
2. The cosine function has the maximum value at 0 and at \( 2\pi \) in \([0, 2\pi]\).
3. The cosine function has the minimum value at \( \pi \) in \([0, 2\pi]\).

Exercise.

1. Prove that \( f(\theta) = \cos \theta \) is symmetrical with respect to \( x = 0 \) in the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

![Graph showing symmetry with respect to \( x = 0 \)]

2. Prove that \( f(\theta) = \cos \theta \) is symmetrical with respect to \( x = \pi \) in the interval \([\frac{\pi}{2}, 3\pi/2]\).

![Graph showing symmetry with respect to \( x = \pi \)]
Example. Identify the amplitude, the period, the phase shift, and sketch the graph of \( f_1(\theta) = 3 \cos(3(\theta + \frac{\pi}{3})) \) over one period.

Answer. Let \( \alpha = 3(\theta + \frac{\pi}{3}) \). Then \( \theta = \frac{\alpha}{3} - \frac{\pi}{3} \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \cos \alpha )</th>
<th>( \frac{\alpha}{3} )</th>
<th>( \theta = \frac{\alpha}{3} - \frac{\pi}{3} )</th>
<th>( \cos 3(\theta + \frac{\pi}{3}) )</th>
<th>( 3 \cos 3(\theta + \frac{\pi}{3}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
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<td>1</td>
<td>3</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
<td>( \frac{\pi}{6} )</td>
<td>( -\frac{\pi}{6} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \pi )</td>
<td>-1</td>
<td>( \frac{\pi}{3} )</td>
<td>0</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>0</td>
<td>( \frac{\pi}{2} )</td>
<td>( \frac{\pi}{6} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2( \pi )</td>
<td>1</td>
<td>( \frac{2\pi}{3} )</td>
<td>( \frac{\pi}{3} )</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
The amplitude of \( f_1 \) is 3, the period is \( \frac{2\pi}{3} \), and the phase shift is \( -\frac{\pi}{3} \).

### 6.3 Tangent Function

We define the tangent number of a given angle \( \theta \), written as \( \tan \theta \) as follows.

\[
\tan \theta = \frac{\sin \theta}{\cos \theta}
\]

Since both \( \sin \theta \) and \( \cos \theta \) are unique numbers for the given \( \theta \), and if \( \cos \theta \neq 0 \), then \( \tan \theta \) is a unique number for a given angle \( \theta \). Also, according to this definition, \( \tan \theta \) does not exist, when \( \cos \theta = 0 \). For example, \( \tan \left( -\frac{\pi}{2} \right) \), \( \tan \left( \frac{\pi}{2} \right) \) or \( \tan \left( \frac{3\pi}{2} \right) \) do not exist.

We can visualize the number \( \tan \theta \) on the unit circle, for given angle \( \theta \) in the domain of the tangent function. First, recall the following theorems from high school.

**The Slope of a Line Theorem.** The slope of a line passing through the points \((x_1, y_1)\) and \((x_2, y_2)\) is \( \frac{y_2 - y_1}{x_2 - x_1} \).

**The Tangent to a Circle Theorem.** A line tangent to a circle is perpendicular to the line passing through the point of tangency and the center of the circle.

Consider an angle \( 0 \leq \theta < \frac{\pi}{2} \). Let \( \ell \) be the line containing the terminal side of the angle \( \theta \) in the standard position. The coordinates of the point of intersection of \( \ell \) and the unit circle is \((\cos \theta, \sin \theta)\). Also \( \ell \) passes through \((0, 0)\). Since we know coordinates of two points on \( \ell \), we can calculate the slope of the line according to the Slope of a Line Theorem. That is, the slope of \( \ell \) is \( \tan \theta \). We can get another point on \( \ell \) as follows. Draw a line \( \ell_1 \) tangent to the unit circle at \((1, 0)\). Let \( P \) be the point of intersection of \( \ell \) and \( \ell_1 \). By the Tangent to a Circle Theorem, \( \ell_1 \perp x\)-axis. Therefore, the \( x \)-coordinate of \( P \) is 1. Suppose the \( y \)-coordinate is \( y_0 \). Then by using \( P(1, y_0) \) and \( O(0, 0) \) to calculate the slope of \( \ell \) we get:

\[
\tan \theta = \frac{y_0 - 0}{1 - 0}.
\]
That is, $y_0 = \tan \theta$.

In other words, the length of the line segment joining $(1, 0)$ and $P$ is $\tan \theta$.

Clearly, $\tan 0 = 0$. We can see now that the tangent number of an angle $\theta$ increases as $\theta$ varies from 0 to $\frac{\pi}{2}$ and the tangent number increases without bounds as $\theta \to \frac{\pi}{2}$.

Now consider an angle $\frac{\pi}{2} < \theta \leq \pi$. For an angle $\theta$ in $(\frac{\pi}{2}, \pi)$, $\sin \theta > 0$ and $\cos \theta < 0$. Therefore, the value of $\tan \theta < 0$ in the interval $(\frac{\pi}{2}, \pi)$. We can visualize the value of the
number $-\tan \theta$, for a given $\theta$ in $(\frac{\pi}{2}, \pi]$ as a length of a line segment just as we did in the interval $[0, \frac{\pi}{2})$.

The value of $-\tan \theta$ decreases as $\theta$ varies from $\frac{\pi}{2}$ to $\pi$. The value of $-\tan \theta$ is 0 when $\theta = \pi$. That means the value of $\tan \theta$ increases in the interval $(\frac{\pi}{2}, \pi]$.

For an angle $\theta$ in $(\pi, \frac{3\pi}{2})$, $\sin \theta < 0$ and $\cos \theta < 0$. Therefore, the value of $\tan \theta > 0$ in the interval $[\pi, \frac{3\pi}{2})$. We can visualize the value of the number $\tan \theta$, for a given $\theta$ in $[\pi, \frac{3\pi}{2})$ as a length of a line segment just as we did before.

The value of $\tan \theta$ increases as $\theta$ varies from $\pi$ to $\frac{3\pi}{2}$. The values of $\tan \theta$ increase without bound when the values of $\theta$ get closer to $\frac{3\pi}{2}$, and $\tan \frac{3\pi}{2}$ does not exist.

For $\theta$ in $(\frac{3\pi}{2}, 2\pi)$, $\sin \theta < 0$ and $\cos \theta > 0$. Therefore, the value of $\tan \theta < 0$ in the interval $(\frac{3\pi}{2}, 2\pi)$. We can visualize the value of the number $-\tan \theta$, for a given $\theta$ in $(\frac{3\pi}{2}, 2\pi]$ as a length of a line segment as before.
The value of $-\tan \theta$ decreases as $\theta$ varies from $\frac{3\pi}{2}$ to $2\pi$. Therefore, the values of $\tan \theta$ increase as $\theta$ varies from $\frac{3\pi}{2}$ to $2\pi$. The values of $\tan \theta$ is 0 when $\theta = 2\pi$.

We can calculate the tangent values of special angles using the definition. The tangent values for special angles are given below. The abbreviation DNE stands for “Does Not Exist”.

<table>
<thead>
<tr>
<th>$\theta$</th>
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</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
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</tr>
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</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$\sqrt{3}$</td>
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<td>DNE</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>$\frac{1}{\sqrt{3}}$</td>
</tr>
<tr>
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<td>$\frac{\sqrt{3}}{3}$</td>
</tr>
<tr>
<td>$\frac{7\pi}{6}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{11\pi}{6}$</td>
<td>$-\frac{1}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
</tr>
</tbody>
</table>

Since $\tan \theta$ is unique for any $\theta$ wherever $\tan \theta$ is defined, $f(\theta) = \tan \theta$ is a function. The domain of this function is all real numbers except where $\cos \theta = 0$.

The collection of all points $(\theta, \tan \theta)$ is called the graph of the function $f(\theta) = \tan \theta$. We can sketch the graph of $f(\theta) = \tan \theta$ by using all the information we have gathered about tangent numbers for $\theta$ in the interval $[0, 2\pi]$. 
There are vertical asymptotes to the tangent function at $\frac{\pi}{2}$ and at $\frac{3\pi}{2}$, by the definition of a vertical asymptote. We will include the vertical asymptotes $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ with the graph of the tangent function in the interval $[0, 2\pi]$ as shown below.

If $\theta = \frac{n\pi}{2}$, for any integer $n$, then $\sin \theta = \pm 1$ and $\cos \theta = 0$. Therefore, by definition, $\tan \theta$
does not exist for such $\theta$. For any other angle $\theta$, the tangent function is defined, since $\cos \theta \neq 0$. That is, the domain of $f(\theta) = \tan \theta$ is all real numbers except the points where $\theta = \frac{n\pi}{2}$, for some integer $n$.

We are going to prove that the tangent function is periodic. We need several “little theorems” first. The word used to describe any little theorem that is needed in a subsequent theorem is “lemma”.

We say a graph of an equation is symmetric with respect to the origin if $(−x, −y)$ is a point on the graph whenever $(x, y)$ is a point on the graph.

**Lemma** (Lemma 1). The unit circle is symmetric with respect to the origin.

**Proof.** The equation of the unit circle is $x^2 + y^2 = 1$. Let $(x_1, y_1)$ be an arbitrary point on the unit circle. Then $x_1^2 + y_1^2 = 1$. We want to show that $(-x_1, -y_1)$ is also a point on the unit circle. This is true because $(-x_1)^2 + (-y_1)^2 = x_1^2 + y_1^2 = 1$.

**Lemma** (Lemma 2). Let $(x_1, y_1)$ be an arbitrary point in the $xy$-coordinate system. Then the line passing through the points $(x_1, y_1)$ and $(-x, -y_1)$ passes through the origin.
Proof. If both $x_1$ and $y_1$ are 0, then $(x_1, y_1) = (−x, −y_1) = (0, 0)$. Any line containing $(0, 0)$ passes through the origin.

If $x_1 = 0$ and $y_1 \neq 0$, then $(0, y_1)$ and $(0, −y_1)$ lies on the horizontal line $y = 0$, and this horizontal line passes through the origin.

If $x_1 \neq 0$ and $y_1 = 0$, then $(x_1, 0)$ and $(-x_1, 0)$ lies on the vertical line $x = 0$, and this vertical line passes through the origin.

If both $x_1$ and $y_1$ are not 0, then the slope of the line $\ell$ passing through $(x_1, y_1)$ and $(-x, -y_1)$ is $\frac{y_1}{x_1}$. Since $\ell$ passes through $(x_1, y_1)$, the equation of $\ell$ is $y - y_1 = \frac{y_1}{x_1}(x - x_1)$. That is, the equation of $\ell$ is $y = \frac{y_1}{x_1}x$. Since $(0, 0)$ is a solution of this equation, $\ell$ passes through the origin.

Theorem. The function $f(\theta) = \tan \theta$ is periodic.

Proof. Let $\theta$ be an arbitrary angle in the domain of the tangent function. First assume that $\theta$ lies in any one of the four quadrants. We know that $P(\cos \theta, \sin \theta)$ is a point on the unit circle. Then by the Lemma 1, $Q(-\cos \theta, -\sin \theta)$ is also a point on the unit circle. By the Lemma 2, The line $PQ$ passes through the origin. Therefore, $−\cos \theta = \cos(\pi + \theta)$ and $−\sin \theta = \sin(\pi + \theta)$. By definition,

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$
6.3. TANGENT FUNCTION

\[
\begin{align*}
\frac{-\sin \theta}{-\cos \theta} &= \frac{\sin(\pi + \theta)}{\cos(\pi + \theta)} \\
&= \tan(\pi + \theta)
\end{align*}
\]

Now suppose that the terminal side of \( \theta \) lies on the \( x \)-axis. Then \( \theta = n\pi \), for some integer \( n \), and \( \sin \theta = 0 \) and \( \cos \theta = \pm 1 \). By definition, \( \tan n\pi = 0 \) for any integer \( n \). In particular, \( \tan(n + 1)\pi = 0 \), since \( n + 1 \) is an integer. Therefore, \( \tan \theta = \tan(\pi + \theta) \).

Since we have chosen \( \theta \) to be an arbitrary angle in the domain of \( f \) in all cases, the function \( f(\theta) = \tan \theta \) is periodic.

\textbf{Theorem.} The period of \( f(\theta) = \tan \theta \) is \( \pi \).

The following is the graph of the tangent function in the interval \([-2\pi, 2\pi]\).
Exercise. Show that the tangent function has no amplitude.

Exercise. Consider the graph of \( f(\theta) = \tan \theta; -\frac{\pi}{2} < \theta < \frac{\pi}{2} \) as the graph of the tangent function over one period. Sketch the graph of \( f_1(\theta) = 2\tan(3(\theta - \frac{\pi}{4})) \) over one period and identify the amplitude, period, and the phase shift.

6.4 Cosecant Function

At this point we recognize that an angle \( \theta \) in radians is just a number. Therefore, we will replace \( \theta \) with \( x \) and sketch the graphs of trigonometric functions in the familiar \( xy \)-coordinate system rather than the \( \theta y \)-coordinate system that we have been using so far.

We define \( \csc x \), pronounced “cosecant x” as

\[
\csc x = \frac{1}{\sin x} \text{ for any real number } x, \text{ where } \sin x \neq 0.
\]

Since \( \sin x \) is unique for any real number \( x \), \( \csc x \) is also unique for any \( x \) where \( \csc x \) is defined. Therefore, \( f(x) = \csc x \) is a function and the domain of \( f \) is all real numbers except where \( \sin x = 0 \).

Since \( \sin x = 0 \), when \( x = n\pi \), for any integer \( n \), the cosecant function \( f(x) = \csc x \) does not exist for \( x = n\pi \). Let us look at \( f(x) = \csc x \) over \([0, 2\pi]\). We will find the cosecant numbers of special angles first.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sin x )</th>
<th>( \csc x )</th>
<th>( x )</th>
<th>( \sin x )</th>
<th>( \csc x )</th>
<th>( x )</th>
<th>( \sin x )</th>
<th>( \csc x )</th>
</tr>
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<td>0</td>
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<td>1</td>
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</tr>
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<td>( 2\pi )</td>
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<td>( -\frac{1}{\sqrt{2}} )</td>
<td>-2</td>
</tr>
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<td>( \sqrt{2} )</td>
<td>( 3\pi )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \sqrt{2} )</td>
<td>( \frac{5\pi}{4} )</td>
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<td>( \frac{2}{\sqrt{3}} )</td>
<td>( 4\pi )</td>
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<tr>
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<td>5\pi</td>
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<td>-2</td>
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<td>-( \sqrt{2} )</td>
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<td>( -\frac{1}{\sqrt{2}} )</td>
<td>-( \sqrt{2} )</td>
</tr>
<tr>
<td>11\pi</td>
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<td>-2</td>
<td>2\pi</td>
<td>0</td>
<td>DNE</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Exercise.

1. Show that as \( x \to 0^+ \), \( \csc x \to \infty \).
2. Show that as \( x \to \pi^- \), \( \csc x \to \infty \).
3. Show that as \( x \to \pi^+ \), \( \csc x \to -\infty \).
4. Show that as \( x \to 2\pi^- \), \( \csc x \to -\infty \).

According to the results of the above exercise, \( x = 0 \), \( x = \pi \) and \( x = 2\pi \) are asymptotes of the cosecant function. With the collected information above, we can sketch the graph of \( f(x) = \csc x \) over the interval \([0, 2\pi]\).

\[ 
\begin{array}{cc}
\text{Exercise.} & \text{Prove the above theorem.} \\
\text{Exercise.} & \text{Show that the function } f(x) = \csc x \text{ has no amplitude.}
\end{array}
\]
6.5 Secant Function

We define sec \( x \), pronounced “secant x” as

\[
\sec x = \frac{1}{\cos x} \quad \text{for any real number } x, \text{ where } \cos x \neq 0.
\]

Since \( \cos x \) is unique for any real number \( x \), sec \( x \) is also unique for any \( x \) where sec \( x \) is defined. Therefore, \( f(x) = \sec x \) is a function, and the domain of \( f \) is all real numbers except where \( \cos x = 0 \).

Since \( \cos x = 0 \), when \( x = \frac{\pm n\pi}{2} \), for any integer \( n \neq 0 \), the secant function \( f(x) = \sec x \) does not exist at those points. Let us look at \( f(x) = \sec x \) over \([0, 2\pi]\). The following are secant numbers of special angles.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos x )</th>
<th>( \sec x )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
</tr>
<tr>
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<td>( \frac{2}{\sqrt{3}} )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
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<td>( \sqrt{2} )</td>
</tr>
<tr>
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<td>( 2 )</td>
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</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos x )</th>
<th>( \sec x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\pi}{2} )</td>
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<td>DNE</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} )</td>
<td>( -\frac{1}{2} )</td>
<td>( -2 )</td>
</tr>
<tr>
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<td>( -\sqrt{2} )</td>
</tr>
<tr>
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<td>( -\frac{\sqrt{3}}{2} )</td>
<td>( -\frac{2}{\sqrt{3}} )</td>
</tr>
<tr>
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<td>( -\frac{\sqrt{3}}{2} )</td>
<td>( -\frac{2}{\sqrt{3}} )</td>
</tr>
<tr>
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</tr>
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<td>DNE</td>
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</tr>
<tr>
<td>( 2\pi )</td>
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<td>1</td>
</tr>
</tbody>
</table>

Exercises.

1. Show that as \( x \to \frac{\pi}{2}^- \), \( \sec x \to \infty \).
2. Show that as \( x \to \frac{\pi}{2}^+ \), \( \csc x \to -\infty \).
3. Show that as \( x \to \frac{\pi}{3}^- \), \( \csc x \to -\infty \).
4. Show that as \( x \to \frac{\pi}{3}^+ \), \( \csc x \to -\infty \).

According to the results of the above exercise, \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \) are asymptotes of the secant function. With the collected information above, we can sketch the graph of \( f(x) = \sec x \) over the interval \([0, 2\pi]\).
Theorem. The function \( f(x) = \sec x \) is periodic, with period \( 2\pi \).

Exercise. Prove the above theorem.

Exercise. Show that the function \( f(x) = \sec x \) has no amplitude.
6.6 Cotangent Function

We define cot \( x \), pronounced “cotangent x” as
\[
\cot x = \frac{1}{\tan x} \quad \text{for any real number} \ x, \text{ where } \tan x \neq 0.
\]

Since \( \tan x \) is unique for any real number \( x \), \( \cot x \) is also unique for any \( x \) where \( \cot x \) is defined.

**Theorem.** \( \cot x = \frac{\cos x}{\sin x} \), for any real number \( x \), where \( \sin x \neq 0 \).

Then \( f(x) = \cot x \) is a function and the domain of \( f \) is all real numbers except where \( \sin x = 0 \).

Since \( \sin x = 0 \), when \( x = n\pi \), for any integer \( n \), the secant function \( f(x) = \cot x \) does not exist at those points. Let us look at \( f(x) = \cot x \) over \([0, 2\pi]\). The following are cotangent numbers of special angles.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \cos x )</th>
<th>( \sin x )</th>
<th>( \cot x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>DNE</td>
</tr>
<tr>
<td>( \pi/6 )</td>
<td>((\sqrt{3}/2))</td>
<td>(1/2)</td>
<td>(\sqrt{3})</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>((1/\sqrt{2}))</td>
<td>((1/\sqrt{2}))</td>
<td>1</td>
</tr>
<tr>
<td>( \pi/3 )</td>
<td>((1/2))</td>
<td>((\sqrt{3}/2))</td>
<td>((1/\sqrt{3}))</td>
</tr>
<tr>
<td>( \pi )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( 5\pi/6 )</td>
<td>(-\sqrt{3}/2)</td>
<td>(1/2)</td>
<td>(-\sqrt{3})</td>
</tr>
<tr>
<td>( 4\pi/3 )</td>
<td>(1/2)</td>
<td>(-\sqrt{3}/2)</td>
<td>(1/\sqrt{3})</td>
</tr>
<tr>
<td>( 3\pi/2 )</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( 5\pi/3 )</td>
<td>((1/2))</td>
<td>(-\sqrt{3}/2)</td>
<td>(-1/\sqrt{3})</td>
</tr>
<tr>
<td>( 7\pi/4 )</td>
<td>((1/\sqrt{2}))</td>
<td>(-1/\sqrt{2})</td>
<td>-1</td>
</tr>
<tr>
<td>( 11\pi/6 )</td>
<td>((\sqrt{3}/2))</td>
<td>(-1/2)</td>
<td>(-\sqrt{3})</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>1</td>
<td>0</td>
<td>DNE</td>
</tr>
</tbody>
</table>
Exercise.

1. Show that as $x \to 0^+$, $\cot x \to \infty$.
2. Show that as $x \to \pi^-$, $\csc x \to -\infty$.
3. Show that as $x \to \pi^+$, $\csc x \to \infty$.
4. Show that as $x \to 2\pi^-$, $\csc x \to -\infty$.

According to the results of the above exercise, $x = 0, x = \pi$ and $x = 2\pi$ are asymptotes of the cotangent function. With the collected information above, we can sketch the graph of $f(x) = \cot x$ over the interval $[0, 2\pi]$.

**Theorem.** The function $f(x) = \cot x$ is periodic with period $\pi$.

**Exercise.** Prove the above theorem.

**Exercise.** Show that the function $f(x) = \cot x$ has no amplitude.
Chapter 7

Inverse Trigonometric Functions

Consider a function $f$ of $x$. For each number $x$ in the domain of $f$, the function $f$ produces a unique number $f(x)$. We can indicate this using the following notation.

$$ f : x \rightarrow f(x) $$

If the number $x$ is also unique, that is, no two numbers $x_1$ and $x_2$ produce $f(x_1) = f(x_2)$, then we can find a function, say $g$, to reverse the process of $f$. That is, for each number $f(x)$, $g$ produces a unique number $x$.

$$ g : f(x) \rightarrow x $$

Not all functions behave this way. For example, consider $f_1(x) = x^2$. Then

$$ f_1 : 2 \rightarrow f(2) = 4 $$

and

$$ f_1 : -2 \rightarrow f(-2) = 4 $$

Therefore, we cannot find a function $g$ to reverse the process of $f_1$. If we restrict the domain of $f_1$ to $[0, \infty)$, then we can reverse the process. Let $f_2 = x^2; x \geq 0$. Then there is a function $g_2$ so that

$$ g_2 : f_2(x) \rightarrow x $$

We say a function with the property “for each number $x$ there is a unique number $f(x)$ and for each $f(x)$ there is a unique number $x$” is “one-to-one” or 1-1.

The following is a theorem that you may have learned in high school.
Theorem. If the graph of a given function $f$ satisfies both the vertical line test and the horizontal line test, then $f$ is 1-1.

In a nutshell, the vertical line test is designed to see if there is a unique $f(x)$ for a given $x$ in the domain of $f$, and the horizontal line test is designed to see if there is a unique $x$ in the domain of $f$ for a given $f(x)$ in the range of $f$.

If a function $f$ is 1-1, then there is a function $g$ so that $g : f(x) \rightarrow x$. We usually use the notation $f^{-1}$, pronounced “eff inverse” for such a function. We call such a function “the inverse function of $f$”. The following is the definition of a inverse function.

We say $f$ and $f^{-1}$ are inverse functions of each other if both of the following two conditions hold.

1. For each $x$ in the domain of $f$, $f^{-1}(f(x)) = x$.
2. For each $x$ in the domain of $f^{-1}$, $f(f^{-1}(x)) = x$.

\begin{center}
\textbf{ab-ba Theorem.} $(a, b)$ is a point on the graph of $f$ if and only if $(b, a)$ is a point on the graph of $f^{-1}$.
\end{center}

\textit{Proof.} Suppose $(a, b)$ is a point on the graph of $f$. Then $b = f(a)$. Since $a$ is on the domain of $f$, $f^{-1}(f(a)) = a$, by definition. That is, $f^{-1}(b) = a$. That is, $(b, a)$ is a point on the graph of $f^{-1}$.

Now suppose $(a, b)$ is a point on the graph of $f^{-1}$. Then $b = f^{-1}(a)$. Since $a$ is on the domain of $f^{-1}$, $f(f^{-1}(a)) = a$, by definition. That is, $f(b) = a$. That is, $(b, a)$ is a point on the graph of $f$. \hfill $\square$

You may have learned the following definition in high school geometry. A point $P(a, b)$ is called the mirror image of the point $Q(c, d)$ across a line $\ell$, and vice versa, if $\ell$ is the perpendicular bisector of the segment $PQ$. 
Let $G_1$ be the graph of a function $f_1$ and let $G_2$ be the graph of a function $f_2$. We say $G_1$ is the mirror image of $G_2$ across a line $\ell$ and vice versa, if each point on $G_1$ is a mirror image of a point on $G_2$ across $\ell$ and each point on $G_2$ is a mirror image of a point on $G_1$ across $\ell$.

**Reflection Theorem.** Let $G_1$ be the graph of a 1-1 function $f$ and let $G_2$ be the graph of $f^{-1}$. Then $G_2$ is the mirror image of $G_1$ across the line $\ell$ whose equation is $y = x$, and vice versa.

**Proof.** We will show that $G_2$ is the mirror image of $G_1$ across $\ell$. Let $P(a, b)$ be an arbitrary point on $G_1$. Then by the ab-ba Theorem, $Q(b, a)$ is on $G_2$. Let $\ell_1$ be the line $L_{PQ}$. Then the slope of $\ell_1$ is $\frac{a-b}{b-a} = -1$. The slope of $\ell$ is $1$. Therefore, $\ell_1$ is perpendicular to $\ell$. The midpoint of $PQ$ is $R(\frac{a+b}{2}, \frac{a+b}{2})$. Clearly, $R$ is on $\ell$. Therefore, $\ell$ is the perpendicular bisector of the segment $PQ$. Since $P$ is arbitrary, $G_2$ is the mirror image of $G_1$ across $\ell$. The proof of the statement “$G_1$ is the mirror image of $G_2$ across $\ell$” is similar.

The graph of the equation $y = x$ is known as the diagonal of the $xy$-plane or simply the diagonal.
7.1 Inverse Sine Function

Let us get the mirror image of the graph of the sine function across the diagonal.

The blue curve in the above figure is the reflection of the graph of the sine function across the diagonal. By using vertical line test, you can clearly see that this graph is not a graph of a function. Therefore, the blue curve cannot be the graph of the inverse sine function. The reason why we cannot use the Reflection Theorem here is that the sine function is not 1-1.

The largest value of the sine function is 1 and the smallest value of the sine function is \(-1\). You get all possible values of the sine function even if we restrict the domain of the sine function to the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

The following is the graph of the restricted sine function \(f(x) = \sin x; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\).

Notice that the restricted sine function is 1-1. Therefore, there is an inverse function for
the restricted sine function. We will use the usual notation reserved for inverse functions to define the inverse sine function.

\[
\sin^{-1}(\sin(x)) = x, \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \text{ and } \\
\sin(\sin^{-1}(x)) = x, \text{ if } -1 \leq x \leq 1.
\]

The blue curve in the following figure is the graph of \( f(x) = \sin^{-1} x \).

7.2 Inverse Cosine Function

Let us get the mirror image of the graph of the cosine function across the diagonal.
The blue curve in the above figure is the reflection of the graph of the cosine function across the diagonal. By using vertical line test, you can clearly see that this graph is not a graph of a function. Therefore, the blue curve cannot be the graph of the inverse cosine function.

We can get all possible values of the sine function even if we restrict the domain of the cosine function to the interval $[0, \pi]$.

The following is the graph of the restricted cosine function $f(x) = \cos x; 0 \leq x \leq \pi$.

Notice that the restricted cosine function is 1-1. Therefore, there is an inverse function for the restricted sine function. We will use the usual notation reserved for inverse functions to define the inverse cosine function.

$$\cos^{-1}(\cos(x)) = x, \text{ if } 0 \leq x \leq \pi,$$ and
\[
\cos(\cos^{-1}(x)) = x, \text{ if } -1 \leq x \leq 1.
\]

The blue curve in the following figure is the graph of \( f(x) = \cos^{-1} x \).

### 7.3 Inverse Tangent Function

Let us get the mirror image of the graph of the tangent function across the diagonal.
The blue curves in the above figure are the reflection of the graph of the tangent function across the diagonal. By using the vertical line test, you can clearly see that this graph is not a graph of a function. Therefore, the blue curves cannot be the graph of the inverse tangent function.

You get all possible values of the tangent function even if we restrict the domain of the tangent function to the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\).

The following is the graph of the restricted tangent function \(f(x) = \tan x; -\frac{\pi}{2} < x < \frac{\pi}{2}\).
Notice that the restricted tangent function is 1-1. Therefore, there is an inverse function for the restricted tangent function. We will use the usual notation reserved for inverse functions to define the inverse tangent function.

\[
\tan^{-1}(\tan(x)) = x, \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \text{ and }
\]

\[
\tan(\tan^{-1}(x)) = x.
\]

The blue curve in the following figure is the graph of \( f(x) = \tan^{-1} x \).
We can similarly define inverse functions for the remaining three trigonometric functions by appropriately restricting the domain of each function. The following is the summary. Filling in the details is left as an exercise.
7.4 Inverse Cosecant Function

\[
csc^{-1}(\csc(x)) = x, \text{ if } x \text{ is in } \left[ -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right] \text{ and }
\]

\[
csc(csc^{-1}(x)) = x, \text{ if } x \text{ is in } (-\infty, -1] \cup [1, \infty).
\]

The blue curve in the following figure is the graph of \( f(x) = \csc^{-1} x \).
7.5 Inverse Secant Function

\[ \sec^{-1}(\sec(x)) = x, \text{ if } x \text{ is in } \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \] and \[ \sec(\sec^{-1}(x)) = x, \text{ if } x \text{ is in } (-\infty, -1] \cup [1, \infty). \]

The blue curve in the following figure is the graph of \( f(x) = \sec^{-1} x \).
7.6 Inverse Cotangent Function

\[ \cot^{-1}(\cot(x)) = x, \text{ if } x \text{ is in } \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \text{ and} \]

\[ \cot(\cot^{-1}(x)) = x. \]

The blue curve in the following figure is the graph of \( f(x) = \cot^{-1} x \).
Example. Find \( \sin^{-1} \left( \sin \frac{5\pi}{6} \right) \), if possible.

Solution. By the definition of the inverse sine function, \( \sin^{-1}(\sin(x)) = x \), if \( x \) is in \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \). Clearly, \( \frac{5\pi}{6} \) is not in \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \). Does that mean \( \sin^{-1} \left( \sin \frac{5\pi}{6} \right) \) doesn’t exist? Notice that \( \sin \frac{5\pi}{6} \) is defined and the sine function is symmetric with respect to the line \( x = \frac{\pi}{2} \).

Therefore, \( \sin \frac{5\pi}{6} = \sin \frac{\pi}{6} \).

Then \( \sin^{-1} \left( \sin \frac{5\pi}{6} \right) = \sin^{-1} \left( \sin \frac{\pi}{6} \right) = \frac{\pi}{6} \), by definition of the inverse sine function.

Example. Find \( \sin \left( \sin^{-1} \left( \frac{5\pi}{6} \right) \right) \), if possible.

Solution. The number \( \frac{5\pi}{6} \) is greater than 1. Therefore, \( \sin^{-1} \left( \frac{5\pi}{6} \right) \) does not exist, by definition. That is, \( \sin \left( \sin^{-1} \left( \frac{5\pi}{6} \right) \right) \) does not exist.

Example. Find \( \sin \left( \sin^{-1} \left( \frac{\pi}{6} \right) \right) \), if possible.

Solution. Clearly, \(-1 < \frac{\pi}{6} < 1\). Therefore, \( \sin^{-1} \left( \frac{5\pi}{6} \right) \) exists and \( \sin \left( \sin^{-1} \left( \frac{\pi}{6} \right) \right) = \frac{\pi}{6} \), by definition.
Chapter 8

Basic Trigonometric Equations

When we looked at quadratic equations in a real number $x$, we realized that we could solve any quadratic equation by completing the square. This observation leads to deriving the quadratic formula for real numbers. In a similar fashion, by knowing that trigonometric functions are periodic and by keeping in mind some symmetries that we observed with trigonometric functions, we should be able to find the solutions to trigonometric equations of the form given below.

1. $\sin x = a$
2. $\cos x = a$
3. $\tan x = a$

where, $a$ is a real number.

Except for one special case, we can solve equations of the form $\csc x = a$, $\sec x = a$ and $\cot x = a$, for a real number $a$, by writing those equations in an equivalent form as $\sin x = \frac{1}{a}$, $\cos x = \frac{1}{a}$ and $\tan x = \frac{1}{a}$. The special case arises when $\cot x = 0$, and we will handle this special case separately.

8.1 Solving the Sine Equation

Consider the equation $\sin x = a$, where $a$ is a real number. First notice that if $a > 1$ or $a < -1$, then this equation has no solutions, as $-1 \leq \sin x \leq 1$, for all real numbers $x$. Therefore, we want to find the solutions of

$\sin x = a$, where $-1 \leq a \leq 1$. 

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First notice that there are infinitely many solutions to this equation. In the following figure ..., $s_{-3}, s_{-2}, s_{-1}, s_1, s_2, s_3, s_4, ...$ are solutions to the above equation for some real number $a$, where $0 < a < 1$.

Consider the case $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Then by the definition of the inverse sine function, 

\[ \sin^{-1}(\sin x) = x. \]

Therefore,

\[ \sin x = a \Rightarrow \sin^{-1}(\sin x) = \sin^{-1} a \Rightarrow x = \sin^{-1} a \]

That is, the solution titled $s_1$ in the above figure is $\sin^{-1} a$.

By using the symmetries that we have discovered in the graph of the sine function, we know that $s_2 = \pi - \sin^{-1} a$. Since the portion of the graph on the interval $[2\pi, 3\pi]$ is congruent to the portion of the graph on the interval $[0, \pi]$ (why?), $s_3 = 2\pi + \sin^{-1} a$ and $s_4 = 3\pi - \sin^{-1} a$. By the same argument, $s_{-1} = -\pi - \sin^{-1} a$, $s_{-2} = -2\pi + \sin^{-1} a$, and $s_{-3} = -3\pi - \sin^{-1} a$, etc. Then the following are the solutions of the trigonometric equation $\sin x = a$ when $0 < a < 1$.

\[ \ldots, (-3\pi - \sin^{-1} a), (-2\pi + \sin^{-1} a), (-\pi - \sin^{-1} a), (\sin^{-1} a), (\pi - \sin^{-1} a), (2\pi + \sin^{-1} a), \ldots \]

**Exercise.** Show that the solutions of $\sin x = a$, where $-1 < a < 0$ are:

\[ \ldots, (-\pi - \sin^{-1} a), (-2\pi + \sin^{-1} a), (\pi - \sin^{-1} a), (\sin^{-1} a), (3\pi - \sin^{-1} a), (2\pi + \sin^{-1} a), \ldots \]

**Exercise.** Show that the solutions of $\sin x = a$, where $a = 0$ are:

\[ \ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots \]
8.1. SOLVING THE SINE EQUATION

The following theorem captures everything we have found so far.

**General Solution to the Sine Equation Theorem.**

Suppose \( \sin x = a \) is a given equation, where \( a \) is a real number.

1. If \( a > 1 \) or \( a < -1 \), then the equation has no solutions.
2. If \( -1 \leq a \leq 1 \), then the general solution of the equation is \( n\pi + (-1)^n \sin^{-1} a \), where \( n \) is any integer.

**Example.** Find the general solution of \( \sin x = -\frac{1}{2} \). Find the solutions of \( \sin x = -\frac{1}{2} \) in the interval \([ -4\pi, 4\pi ] \).

**Solution.** By the General Solution to the Sine Equation Theorem, the solutions are:

\[
x = n\pi + (-1)^n \sin^{-1} \left( -\frac{1}{2} \right), \quad \text{for any integer } n.
\]

Since \( \sin \left( -\frac{\pi}{6} \right) = -\frac{1}{2} \), by the definition of the inverse sine function, \( \sin^{-1} \left( -\frac{1}{2} \right) = \sin^{-1} \left( \sin \left( -\frac{\pi}{6} \right) \right) = -\frac{\pi}{6} \). Therefore, the general solution to the given equation is:

\[
x = n\pi - (-1)^n \frac{\pi}{6}, \quad \text{for any integer } n.
\]

When \( n = 0 \), \( x = -\frac{\pi}{6} \). This solution is more than \(-4\pi \). We keep it.

When \( n = 1 \), \( x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \). This solution is less than \( 4\pi \). We keep it.

When \( n = 2 \), \( x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \). This solution is less than \( 4\pi \). We keep it.

When \( n = 3 \), \( x = 3\pi + \frac{\pi}{6} = \frac{19\pi}{6} \). This solution is less than \( 4\pi \). We keep it.

When \( n = 4 \), \( x = 4\pi - \frac{\pi}{6} = \frac{23\pi}{6} \). This solution is less than \( 4\pi \). We keep it.

When \( n = 5 \), \( x = 5\pi + \frac{\pi}{6} \). This solution is more than \( 4\pi \), and any larger value of \( n \) produces a solution larger than \( 4\pi \). Therefore, we will not keep solutions for any \( n \) more than 4.

Now let us try negative integers.

When \( n = -1 \), \( x = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6} \). This solution is more than \(-4\pi \). We keep it.

When \( n = -2 \), \( x = -2\pi - \frac{\pi}{6} = -\frac{13\pi}{6} \). This solution is more than \(-4\pi \). We keep it.

When \( n = -3 \), \( x = -3\pi + \frac{\pi}{6} = -\frac{17\pi}{6} \). This solution is more than \(-4\pi \). We keep it.

When \( n = -4 \), \( x = -4\pi - \frac{\pi}{6} \). This solution is less than \(-4\pi \), and any smaller value of
n produces a solution less than $-4\pi$. Therefore, we will not keep solutions for any $n$ less than $-3$.

Therefore, the solutions of the equation $\sin x = -\frac{1}{2}$ in the interval $[-4\pi, 4\pi]$ are:

\[-\frac{17\pi}{6}, -\frac{13\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}\]

\[\square\]

The following figure shows the solutions to $\sin x = -\frac{1}{2}$ in the interval $[-4\pi, 4\pi]$ graphically.

---

**Example.** Find the general solution of $\sin 3x = \frac{\sqrt{3}}{2}$. Find the solutions of $\sin 3x = \frac{\sqrt{3}}{2}$ in the interval $[-\pi, \pi]$.

**Solution.** By the General Solution to the Sine Equation Theorem, the solutions are:

$$3x = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right), \text{ for any integer } n.$$

Since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, by the definition of the inverse sine function, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}(\sin\left(\frac{\pi}{3}\right)) = \frac{\pi}{3}$. Therefore,

$$3x = n\pi + (-1)^n \frac{\pi}{3}, \text{ for any integer } n.$$

By multiplying both sides by $\frac{1}{3}$, we get the general solution to the given equation.

$$x = \frac{1}{3}n\pi + (-1)^n \frac{\pi}{9}, \text{ for any integer } n. \; \square$$

When $n = 0$, $x = \frac{\pi}{9}$. This solution is less than $\pi$. We keep it.

When $n = 1$, $x = \frac{\pi}{3} - \frac{\pi}{9} = \frac{2\pi}{9}$. This solution is less than $\pi$. We keep it.

When $n = 2$, $x = \frac{2\pi}{3} + \frac{\pi}{9} = \frac{7\pi}{9}$. This solution is less than $\pi$. We keep it.

When $n = 3$, $x = \pi - \frac{\pi}{9} = \frac{8\pi}{9}$. This solution is less than $\pi$. We keep it.

When $n = 4$, $x = \frac{4\pi}{3} + \frac{\pi}{9} = \frac{13\pi}{9}$. This solution is more than $\pi$, and any larger value of $n$ produces a solution larger than $\pi$. Therefore, we will not keep solutions for any $n$ more than 4.

Now let us try negative integers.

When $n = -1$, $-\frac{\pi}{3} - \frac{\pi}{9} = -\frac{4\pi}{9}$. This solution is more than $-\pi$. We keep it.
8.2. SOLVING THE COSINE EQUATION

When \( n = -2, -\frac{2\pi}{3} + \frac{\pi}{9} = -\frac{5\pi}{9} \). This solution is more than \(-\pi\). We keep it.
When \( n = -3, x = -\pi - \frac{\pi}{9} = -\frac{10\pi}{6} \). This solution is less than \(-\pi\), and any smaller value of \( n \) produces a solution less than \(-\pi\). Therefore, we will not keep solutions for any \( n \) less than \(-3\).

Therefore, the solutions of the equation \( \sin 3x = \frac{\sqrt{3}}{2} \) in the interval \([-\pi, \pi]\) are:

\[-\frac{5\pi}{9}, -\frac{4\pi}{9}, \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9} \]

\[\square\]

8.2 Solving the Cosine Equation

Consider the equation \( \cos x = a \), where \( a \) is a real number. First notice that if \( a > 1 \) or \( a < -1 \), then this equation has no solutions, as \(-1 \leq \cos x \leq 1\), for all real numbers \( x \). Therefore, we want to find the solutions of

\[ \cos x = a, \text{ where } -1 \leq a \leq 1. \]

Again notice that there are infinitely many solutions to this equation. In the following figure . . . , \( s_{-3}, s_{-2}, s_{-1}, s_1, s_2, s_3, s_4, . . . \) are solutions to the above equation for some real number \( a \), where \( 0 < a < 1 \).

Consider the case \( 0 \leq x \leq \pi \). Then by the definition of the inverse cosine function, \( \cos^{-1}(\cos x) = x \). Therefore,

\[ \cos x = a \]

\[ \implies \cos^{-1}(\cos x) = \cos^{-1} a \]

\[ \implies x = \cos^{-1} a \]

That is, the solution titled \( s_1 \) in the above figure is \( \cos^{-1} a \).
By using the symmetries that we have discovered in the graph of the cosine function, we know that $s_{-1} = -\cos^{-1} a$. Since the portion of the graph on the interval $[\frac{3\pi}{2}, \frac{5\pi}{2}]$ is congruent to the portion of the graph on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (why?), $s_{2} = 2\pi - \cos^{-1} a$ and $s_{3} = 2\pi + \cos^{-1} a$. By the same argument, $s_{-2} = -2\pi + \cos^{-1} a$ and $s_{-3} = -2\pi - \cos^{-1} a$, etc. Then the following are the solutions of the trigonometric equation $\cos x = a$ when $0 < a < 1$ are:

$$\ldots, (-2\pi - \cos^{-1} a), (-2\pi + \cos^{-1} a), (-\cos^{-1} a), (\cos^{-1} a), (2\pi - \cos^{-1} a), (2\pi + \cos^{-1} a), \ldots$$

**Exercise.** Show that the solutions of $\cos x = a$, where $-1 < a < 0$.

$$\ldots, (-2\pi - \cos^{-1} a), (-2\pi + \cos^{-1} a), (-\cos^{-1} a), (\cos^{-1} a), (2\pi - \cos^{-1} a), (2\pi + \cos^{-1} a), \ldots$$

**Exercise.** Show that the solutions of $\cos x = a$, where $a = 0$ are:

$$\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

The following theorem captures everything we have found so far.

**General Solution to the Cosine Equation Theorem.**

*Suppose $\cos x = a$ is a given equation, where $a$ is a real number.*

1. *If $a > 1$ or $a < -1$, then the equation has no solutions.*
2. *If $-1 \leq a \leq 1$, then the general solution of the equation is $2n\pi \pm \cos^{-1} a$, where $n$ is any integer.*

**Example.** Find the general solution of $\cos x = -\frac{1}{2}$. Find the solutions of $\cos x = -\frac{1}{2}$ in the interval $[-4\pi, 4\pi]$.

**Solution.** By the General Solution to the Cosine Equation Theorem, the solutions are:

$$x = 2n\pi \pm \cos^{-1} \left(-\frac{1}{2}\right), \text{ for any integer } n.$$  

Since $\cos \left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, by the definition of the inverse sine function, $\cos^{-1} \left(-\frac{1}{2}\right) = \cos^{-1}(\cos \left(\frac{2\pi}{3}\right)) = \frac{2\pi}{3}$. Therefore, the general solution the given equation is:

$$x = 2n\pi \pm \frac{2\pi}{3}, \text{ for any integer } n. \quad \square$$
When $n = 0$, $x = -\frac{2\pi}{3}$ or $x = \frac{2\pi}{3}$. Both answers are in the interval $[-4\pi, 4\pi]$ so we keep both of them.

When $n = 1$, $x = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$ or $x = 2\pi + \frac{2\pi}{3} = \frac{8\pi}{3}$. Both answers are in the interval $[-4\pi, 4\pi]$ so we keep both of them.

When $n = 2$, $x = 4\pi - \frac{2\pi}{3} = \frac{10\pi}{3}$ or $x = 4\pi + \frac{2\pi}{3}$. We keep the first solution but discard the second solution, as it is more than $4\pi$.

When $n = 3$, $x = 6\pi - \frac{2\pi}{3}$ or $x = 6\pi + \frac{2\pi}{3}$. Both of these solutions are more than $4\pi$. Therefore, we will not keep solutions for any $n$ more than $3$.

Now let us try negative integers.

When $n = -1$, $x = -2\pi - \frac{2\pi}{3} = -\frac{8\pi}{3}$ or $x = -2\pi + \frac{2\pi}{3} = -\frac{4\pi}{3}$. Both answers are in the interval $[-4\pi, 4\pi]$ so we keep both of them.

When $n = -2$, $x = -4\pi - \frac{2\pi}{3}$ or $x = -4\pi + \frac{2\pi}{3} = -\frac{10\pi}{3}$. We keep the second solution but discard the first solution, as it is less than $-4\pi$.

When $n = -3$, $x = -6\pi - \frac{2\pi}{3}$ or $x = -6\pi + \frac{2\pi}{3}$. Both of these solutions are less than $-4\pi$. Therefore, we will not keep solutions for any $n$ less than $-3$.

The solutions of the equation $\cos x = -\frac{1}{2}$ in the interval $[-4\pi, 4\pi]$ are:

$$-\frac{10\pi}{3}, -\frac{8\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

The following figure shows the solutions to $\cos x = -\frac{1}{2}$ in the interval $[-4\pi, 4\pi]$ graphically.

8.3 Solving the Tangent Equation

Consider the equation $\tan x = a$, where $a$ is a real number.

Again notice that there are infinitely many solutions to this equation. In the following figure $\ldots, s_{-3}, s_{-2}, s_{-1}, s_1, s_2, s_3, s_4, \ldots$ are solutions to the above equation for some real number $a$. 
Consider the case $-\frac{\pi}{2} < x < \frac{\pi}{2}$. By the definition of the inverse tangent function, $\tan^{-1}(\tan x) = x$. Therefore,

\[
\tan x = a \\
\implies \tan^{-1}(\tan x) = \tan^{-1} a \\
\implies x = \tan^{-1} a
\]

That is, the solution titled $s_1$ in the above figure is $\tan^{-1} a$. 
By using the symmetries that we have discovered in the graph of the tangent function, we know that \( s_1 = \pi + \tan^{-1} a \). Since the portion of the graph on the interval \( \left( \frac{3\pi}{2}, \frac{5\pi}{2} \right) \) is congruent to the portion of the graph on the interval \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), \( s_2 = 2\pi + \tan^{-1} a \) and \( s_3 = 3\pi + \tan^{-1} a \). By the same argument, \( s_{-1} = -\pi + \tan^{-1} a \) and \( s_{-2} = -2\pi + \tan^{-1} a \), etc. Then the following are the solutions of the trigonometric equation \( \tan x = a \) when \( a > 0 \) are:

\[
\ldots, (-3\pi + \tan^{-1} a), (-2\pi + \tan^{-1} a), (\pi - \tan^{-1} a), (\tan^{-1} a), (\pi + \tan^{-1} a), (2\pi + \tan^{-1} a), \ldots
\]

**Exercise.** Show that the solutions of \( \cos x = a \), where \( a < 0 \) are:

\[
\ldots, (-3\pi + \tan^{-1} a), (-2\pi + \tan^{-1} a), (\pi - \tan^{-1} a), (\tan^{-1} a), (\pi + \tan^{-1} a), (2\pi + \tan^{-1} a), \ldots
\]

**Exercise.** Show that the solutions of \( \tan x = a \), where \( a = 0 \) are:

\[
\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots
\]

The following theorem captures everything we have found so far.
**General Solution to the Tangent Equation Theorem.**

Suppose $\tan x = a$ is a given equation, where $a$ is a real number.

Then the general solution of the equation is $n\pi + \tan^{-1} a$, where $n$ is any integer.

**Example.** Find the general solution of $\tan x = -1$. Find the solutions of $\tan x = -1$ in the interval $[-2\pi, 2\pi]$.

**Solution.** By the General Solution to the Tangent Equation Theorem, the solutions are:

$$x = n\pi + \tan^{-1} (-1), \text{ for any integer } n.$$  

Since $\tan \left( -\frac{\pi}{4} \right) = -1$, by the definition of the inverse sine function, $\tan^{-1}(-1) = \tan^{-1}(\tan \left( -\frac{\pi}{4} \right)) = -\frac{\pi}{4}$. Therefore, the general solution the given equation is:

$$x = n\pi - \frac{\pi}{4}, \text{ for any integer } n. \quad \square$$

When $n = 0$, $x = -\frac{\pi}{4}$. This solution is in the interval $[-2\pi, 2\pi]$ so we keep it.

When $n = 1$, $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. This solution is in the interval $[-2\pi, 2\pi]$ so we keep it.

When $n = 2$, $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$. This solution is in the interval $[-2\pi, 2\pi]$ so we keep it.

When $n = 3$, $x = 3\pi - \frac{\pi}{4}$ or $x = 2\pi + \frac{3\pi}{4}$. This solution is more than $2\pi$. Therefore, we will not keep solutions for any $n$ more than 2.

Now let us try negative integers.

When $n = -1$, $x = -\pi - \frac{\pi}{4} = -\frac{5\pi}{4}$. This solution is in the interval $[-2\pi, 2\pi]$ so we keep it.

When $n = -2$, $x = -2\pi - \frac{\pi}{4}$. This solution is less than $2\pi$. Therefore, we will not keep solutions for any $n$ less than 1.

The solutions of the equation $\tan x = -1$ in the interval $[-2\pi, 2\pi]$ are:

$$-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4} \quad \square$$

The following figure shows the solutions to $\tan x = -1$ in the interval $[-2\pi, 2\pi]$ graphically.
Now we will look at the special case mentioned at the beginning of this section. That is how to solve the cot \( x = 0 \). In the domain of the inverse cotangent function, that is, in the intervals \((-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\), cot \( x = 0 \) when \( x = \frac{\pi}{2} \). Therefore, \( \cot^{-1}(0) = \frac{\pi}{2} \), by the definition of the inverse cotangent function. You should be able to easily prove the following theorem now.

**Theorem.** The general solution to the equation cot \( x = 0 \) is \( n\pi + \frac{\pi}{2} \), for any integer \( n \).
In this chapter we will use the degree as the unit for the measure of an angle. We will also use the following conventions throughout this section. Consider a given triangle $ABC$. What we mean by this is that the vertices of the triangle are identified by the letters $A$, $B$, and $C$. The angle at the vertex $A$ is identified as $\alpha$, the angle at vertex $B$ is identified as $\beta$, and the angle at the vertex $C$ is identified as $\gamma$. The side facing the angle $\alpha$ is identified as $a$, the side facing the angle $\beta$ is identified as $b$, and the side facing the angle $\gamma$ is identified as $c$.

We will refer to a triangle with angles and sides identified according to the above conventions as a triangle with standard notations.

The phrase “solving a triangle” means finding all six quantities; namely, the measures of angles $\alpha$, $\beta$, and $\gamma$, and the measures of the sides $a$, $b$, and $c$. We will loosely use the names of the angles for angle measures as well. That is, it is understood that by saying “$\alpha$ is 60 degrees” we really mean “the angle $\alpha$ has a measure of 60 degrees”. We will also
loosely use the name of a side of a triangle for its measure. That is, by saying “a is 5 units” we really mean that “the length of the side a is 5 units”.

The following theorems on triangles that you may have heard about in middle school and may have proved in high school will be useful throughout this section.

**Angle Sum of a Triangle Theorem.** *The sum of the degrees of the angles of a triangle is $180^\circ$.***

**Pythagorean Theorem.** *Suppose $\gamma = 90^\circ$. Then $a^2 + b^2 = c^2$.***

**Converse of the Pythagorean Theorem.** *Suppose $a^2 + b^2 = c^2$. Then $\gamma = 90^\circ$.***

### 9.1 Solving a Right Triangle

Consider a right triangle. That is, one of the angles is a right angle. Then by the Angle Sum of a Triangle Theorem, the other two angles are acute.

Let us name the vertices of the given right triangle as follows.
Then according to our convention, the side $AB$ is $c$, the side $BC$ is $a$ and the side $AC$ is $b$, and the angle $B$ is $\gamma$.

Introduce a coordinate system and move the triangle $ABC$ by using rigid motions: rotate, translate, or reflect, so that $B$ is at the origin, and $BC$ is on the positive $x$ axis. Since $\beta$ is acute, the point $A$ lies in the first quadrant and the coordinates of $A$ are $(a,b)$.

Notice that the angle $\beta$ is in the standard position and $A$ is a point on the terminal side of $\beta$. Therefore, by the definition of the sine number of $\beta$, $\sin \beta = \frac{b}{c}$, and by the definition of a cosine number of $\beta$, $\cos \beta = \frac{a}{c}$. Therefore, $a = c \cos \beta$ and $b = c \cos \beta$. 
Now if we remove the coordinate system then we have the following information for the given right triangle.

So, for a right triangle, we can think of the measure of the adjacent side of $\beta$ as $c \cos \beta$ and the length of the opposite side as $c \sin \beta$.

Combining the above observations, we have the following theorem. We will deviate from our convention just for this theorem only to demonstrate that the theorem does not depend on the convention.

**Trigonometric Numbers of a Triangle Theorem.** Consider a right triangle and let $\theta$ be one of the acute angles. Then

1. $\sin \theta = \frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse}}$
2. \( \cos \theta = \frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse}} \)

3. \( \tan \theta = \frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}} \)

**Proof.** Let \( ABC \) be the given triangle and let \( \theta \) be the angle \( B \) as shown. Let \( AB = c \).

Then \( BC = c \cos \theta \) and \( AC = c \sin \theta \).

Therefore,

\[
\frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse}} = \frac{|AC|}{|AB|} = \frac{c \sin \theta}{c} = \sin \theta.
\]

\[
\frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse}} = \frac{|BC|}{|AB|} = \frac{c \cos \theta}{c} = \cos \theta.
\]

\[
\frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}} = \frac{|AC|}{|BC|} = \frac{c \sin \theta}{c \cos \theta} = \tan \theta.
\]

\( \square \)

**Theorem.** A right triangle can be solved if a length of a side and the measure of one acute angle are given.
Exercise. Prove the above theorem.

Example. Suppose the length of the hypotenuse of a right triangle is 13 units and one of the acute angles is $31^\circ$. Solve the triangle.

Answer. Suppose the given triangle is $ABC$ as shown below.

We know that $c = 13$, $\beta = 31^\circ$, and $\gamma = 90^\circ$. We have to find $\alpha$, $a$, and $b$ to solve this triangle.

By the Angle Sum of a Triangle Theorem, $\alpha + 31 + 90 = 180$, all in degrees. Therefore, $\alpha = 59^\circ$. $b = 13 \sin 31^\circ$. If we want to know this number approximately then we could use a calculator. With the help of a calculator, $b \approx 6.7$ units. $a = 13 \cos 31^\circ$. With the help of a calculator, $a \approx 11.1$ units. Therefore, $a \approx 11.1$ length units, $b \approx 6.7$ length units, $c = 13$ units, $\alpha = 59^\circ$, $\beta = 31^\circ$ and $\gamma = 90^\circ$.

Theorem. A right triangle can be solved if the lengths of two sides are given.

Exercise. Prove the above theorem.

Example. Suppose the length of the hypotenuse of a right triangle is 13 length units and one of the other two sides is 11 length units. Solve the triangle.
Answer. Suppose the given triangle is ABC as shown below.

We know that \( c = 13 \), \( a = 11 \), and \( \gamma = 90^\circ \). We have to find \( \alpha \), \( \beta \), and \( b \) to solve this triangle.

By the Pythagorean Theorem, \( 13^2 = 11^2 + b^2 \). Therefore, \( b^2 = 48 \) and \( b = \pm \sqrt{48} \). Since \( b \) is a length, \( b > 0 \). Therefore, \( b = 4\sqrt{3} \).

By the Trigonometric Numbers of a Triangle Theorem, \( \cos \beta = \frac{11}{13} \). Notice that this is a trigonometric equation. The general solution is \( 2n\pi \pm \cos^{-1}\left(\frac{11}{13}\right) \), for any integer \( n \). Since \( \beta \) is acute \( \beta \) is in \( (0, \frac{\pi}{2}) \). The inverse cosine function is defined in this interval, and among all solutions of \( \cos \beta = \frac{11}{13} \), the only solution in \( (0, \frac{\pi}{2}) \) is \( \cos^{-1}\left(\frac{11}{13}\right) \). Therefore, \( \beta = \cos^{-1}\left(\frac{11}{13}\right) \).

By using a calculator we can find an approximation of this number. Accordingly, \( \beta \approx 32^\circ \).

By the Sum of The Angles of a Triangle Theorem, \( \alpha \approx 180^\circ - (90^\circ + 32^\circ) \) in degrees. Therefore, \( \alpha = 58^\circ \).

Now we have all six quantities. \( a = 11 \) length units, \( b = 4\sqrt{3} \) length units, \( c = 13 \) length units, \( \alpha = 58^\circ \), \( \beta = 32^\circ \), and \( \gamma = 90^\circ \).
9.2 Solving Triangles that are not Right Triangles

Consider a triangle which is not a right triangle.

It is important to know that the Pythagorean Theorem is not available any more. However, the Sum of the Angles of a Triangle Theorem is still useful. In addition, the following theorems that you may have learned in high school can be useful.

**Triangle Inequality Theorem.** In a triangle, sum of the lengths of two sides is longer than the length of the third side.

**Longer Side of a Triangle Theorem.** In a triangle, a side facing a larger angle is longer.

For example, suppose $\alpha > \gamma$. (See the above figure.) Then $a > c$.

**Larger Angle of a Triangle Theorem.** In a triangle, an angle facing a longer side is larger.

For example, suppose $c > b$. (See the above figure.) Then $\gamma > \beta$. 
9.2. SOLVING TRIANGLES THAT ARE NOT RIGHT TRIANGLES

ASA Theorem. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$.

1. If $\alpha_1 = \alpha_2$, $c_1 = c_2$, and $\beta_1 = \beta_2$, then the two triangles are congruent.
2. If $\beta_1 = \beta_2$, $a_1 = a_2$, and $\gamma_1 = \gamma_2$, then the two triangles are congruent.
3. If $\gamma_1 = \gamma_2$, $b_1 = b_2$, and $\alpha_1 = \alpha_2$, then the two triangles are congruent.

SAS Theorem. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$.

1. If $b_1 = b_2$, $\alpha_1 = \alpha_2$, and $c_1 = c_2$, then the two triangles are congruent.
2. If $c_1 = c_2$, $\beta_1 = \beta_2$, and $a_1 = a_2$, then the two triangles are congruent.
3. If $a_1 = a_2$, $\gamma_1 = \gamma_2$, and $b_1 = b_2$, then the two triangles are congruent.

SSS Theorem. Consider two triangles $A_1B_1C_1$ and $A_2B_2C_2$.
If $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$, then the two triangles are congruent.

Cross Multiplication Algorithm Theorem. If $A$, $B$, $C$ and $D$ are real numbers and $B \neq 0$ and $D \neq 0$, then

$$AD = BC \text{ if and only if } \frac{A}{B} = \frac{C}{D}.$$
The Law of Sines Theorem. Let \(ABC\) be a triangle with the standard notations. Then

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.
\]

That is, we have to show that \(\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}\), and \(\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}\) to prove this theorem. Since the equality is transitive for real numbers, if we can prove the first two then the third follows. Any angle (measure) of a triangle lies in the interval \((0, \pi)\). Therefore, \(\sin \alpha \neq 0\), \(\sin \beta \neq 0\), and \(\sin \gamma \neq 0\). Clearly, \(a \neq 0\), \(b \neq 0\), and \(c \neq 0\). Then by using the Cross Multiplication Algorithm Theorem, we also get:

\[
\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.
\]

We will prove \(\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}\) and leave the proof of the other half as an exercise.

Proof. Drop a perpendicular from \(A\) to the side \(BC\). Let \(D\) be the foot of this perpendicular. There are two cases. The point \(D\) lies between \(B\) and \(C\) or \(D\) lies outside of \(BC\).

In the first case, from the right triangle \(ABD\), we get \(|AD| = c \sin \beta\), and from the triangle \(ACD\), we get \(|AD| = b \sin \gamma\). Therefore,

\[c \sin \beta = b \sin \gamma.\]
By using the Cross Multiplication Algorithm,\[
\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.
\]

In the second case, let us assume that $D$ lies on the extended segment $BC$ as shown in the figure. Then from the right triangle $ABD$ we get $|AD| = c \sin \beta$ and from the triangle $ACD$ we get $|AD| = b \sin(180^\circ - \gamma)$. Since the sine function is symmetrical with respect to the line $x = \frac{\pi}{2}$, $\sin(180^\circ - \gamma) = \sin \gamma$. Therefore,
\[
c \sin \beta = b \sin \gamma.
\]

By using the Cross Multiplication Algorithm Theorem,
\[
\frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.
\]

\[\square\]

**Exercise.** Let $ABC$ be a triangle with standard notations. Show that $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$.

**Example.** Let $ABC$ be the triangle with standard notations. Suppose $\beta = 20^\circ$, $a = 7$ units, $\gamma = 40^\circ$. Solve the triangle, if possible.

**Answer.** First notice that we can easily construct a triangle with the given properties by using a ruler and a compass. That is, a triangle with the given properties exists. By the ASA Theorem, there is only one such triangle up to a congruence. That is, any other triangle with these properties is congruent to this triangle. We have to find $\alpha$, $b$ and $c$ to solve the triangle.
By the Sum of the Angles of a Triangle Theorem, $\alpha = 120^\circ$. By the Law of Sines Theorem,

$$\frac{b}{\sin 20^\circ} = \frac{7}{\sin 120^\circ}.$$

By using the Cross Multiplication Algorithm Theorem,

$$b = \frac{7 \sin 20^\circ}{\sin 120^\circ}.$$

By using a calculator, $b \approx 2.7$ units. By the Law of Sines Theorem,

$$\frac{c}{\sin 40^\circ} = \frac{7}{\sin 120^\circ}.$$

By using the Cross Multiplication Algorithm Theorem,

$$c = \frac{7 \sin 40^\circ}{\sin 120^\circ}.$$

By using a calculator, $c \approx 5.2$ units.

Now we have all six quantities. $a = 7$ length units, $b \approx 2.7$ length units, $c \approx 5.2$ length units, $\alpha = 120^\circ$, $\beta = 20^\circ$ and $\gamma = 40^\circ$.

**Example.** Let $ABC$ be the triangle with standard notations. Suppose $\beta = 120^\circ$, $a = 7$ units, $\gamma = 70^\circ$. Solve the triangle, if possible.

**Answer.** Since $120^\circ + 70^\circ > 180^\circ$, there is no such triangle, by the Angle Sum of a Triangle Theorem.

**Example.** Let $ABC$ be the triangle with standard notations. Suppose $\beta = 20^\circ$, $a = 7$ units, $b = 3$ units. Solve the triangle, if possible.

**Answer.** With the result of the previous example, we know that it is possible that there may be no triangle with the given properties. Instead of trying to construct a triangle with the given properties, let us assume that the given triangle exists and use known theorems,
and try to solve the triangle. In the process, if we encounter a contradiction, then we know that our assumption is false and there can be no such triangle.

So, let us assume that there is such a triangle.

First we can find $\alpha$ by using the Law of Sines Theorem.

\[
\frac{\sin \alpha}{7} = \frac{\sin 20^\circ}{3}.
\]

By the Cross Multiplication Algorithm Theorem,

\[
\sin \alpha = \frac{7 \sin 20^\circ}{3}.
\]

An angle of a triangle is greater than $0^\circ$ and less than $180^\circ$. That is, we have a sine equation in $\alpha$, and we want to find the solutions in the interval $(0^\circ, 180^\circ)$. The general solution (in degrees) is:

\[
\alpha = n(180^\circ) + (-1)^n \sin^{-1}\left(\frac{7 \sin 20^\circ}{3}\right), \text{ where } n \text{ is any integer.}
\]

when $n = 0$, we get,

\[
\alpha = \sin^{-1}\left(\frac{7 \sin 20^\circ}{3}\right)
\]

and when $n = 1$, we get,

\[
\alpha = 180^\circ - \sin^{-1}\left(\frac{7 \sin 20^\circ}{3}\right)
\]

It would be hard to proceed without knowing approximately what $\sin^{-1}\left(\frac{7 \sin 20^\circ}{3}\right)$ is. We will use a calculator to get an approximation of this quantity. It turns out,

\[
\sin^{-1}\left(\frac{7 \sin 20^\circ}{3}\right) \approx 53^\circ.
\]
Therefore, \( \alpha \approx 53^\circ \) or \( \alpha \approx 127^\circ \).

**case 1:** \( \alpha \approx 53^\circ \).

*By the Sum of the Angles of a Triangle Theorem, \( \gamma = 107^\circ \). By the Law of Sines Theorem,*

\[
\frac{c}{\sin \gamma} = \frac{3}{\sin 20^\circ}.
\]

*By the Cross Multiplication Algorithm theorem,*

\[
c = \frac{3 \sin \gamma}{\sin 20^\circ}.
\]

*By using a calculator, \( c \approx 8.4 \) units.*

**case 2:** \( \alpha \approx 127^\circ \).

*By the Sum of the Angles of a Triangle Theorem, \( \gamma = 33^\circ \). By the Law of Sines Theorem,*

\[
\frac{c}{\sin \gamma} = \frac{3}{\sin 20^\circ}.
\]

*By the Cross Multiplication Algorithm theorem,*

\[
c = \frac{3 \sin \gamma}{\sin 20^\circ}.
\]

*By using a calculator, \( c \approx 4.8 \) units.*

There are two triangles with the given properties.

1. \( a = 7, \ b = 3, \ c \approx 8.4, \alpha \approx 53^\circ, \beta = 20^\circ, \gamma \approx 107^\circ. \)
2. \( a = 7, \ b = 3, \ c \approx 4.8, \alpha \approx 127^\circ, \beta = 20^\circ, \gamma \approx 33^\circ \)

**Example.** Let \( ABC \) be the triangle with standard notations. Suppose \( \beta = 80^\circ, \ a = 7 \) units, \( b = 3 \) units. Solve the triangle, if possible.

**Answer.** Suppose there is a triangle with the given properties.
By the Law of Sines Theorem,
\[ \frac{\sin \alpha}{7} = \frac{\sin 80^\circ}{3}. \]

By the Cross Multiplication Algorithm,
\[ \sin \alpha = \frac{7 \sin 80^\circ}{3}. \]

That is,
\[ \sin \alpha \approx 2.29788 \]

This is impossible since \(-1 \leq \sin \alpha \leq 1\). Therefore, our assumption is false. That is, there is no such triangle.

Example. Let \( ABC \) be the triangle with standard notations. Suppose \( \beta = 87^\circ, a = 6.8 \) units, \( b = 7 \) units. Solve the triangle, if possible.

Answer. Suppose there is a triangle with the given properties.

\[\begin{array}{c}
A \\
c \\
\alpha \\
7 \\
\beta \\
87^\circ \\
6.8 \\
B \\
\gamma \\
C
\end{array}\]

By the Law of Sines Theorem,
\[ \frac{\sin \alpha}{6.8} = \frac{\sin 87^\circ}{7}. \]

By the Cross Multiplication Algorithm,
\[ \sin \alpha = \frac{6.8 \sin 87^\circ}{7}. \]

The general solution to this equation is:
\[ \alpha = n(180^\circ) + (-1)^n \sin^{-1} \left( \frac{6.8 \sin 87^\circ}{7} \right), \text{ where } n \text{ is any integer.} \]
when \( n = 0 \), we get,

\[
\alpha = \sin^{-1}\left(\frac{6.8 \sin 87^\circ}{7}\right) \approx 76^\circ
\]

and when \( n = 1 \), we get,

\[
\alpha = 180^\circ - \sin^{-1}\left(\frac{6.8 \sin 87^\circ}{7}\right) \approx 104^\circ
\]

**case 1:** \( \alpha \approx 76^\circ \).

By the Sum of the Angles of a Triangle Theorem, \( \gamma = 17^\circ \). By the Law of Sines Theorem,

\[
\frac{c}{\sin \gamma} = \frac{3}{\sin 20^\circ}.
\]

By the Cross Multiplication Algorithm theorem,

\[
c = \frac{3 \sin \gamma}{\sin 20^\circ}.
\]

By using a calculator, \( c \approx 2.05 \) units.

**case 2:** \( \alpha \approx 104^\circ \).

Since \( 87^\circ + 104^\circ > 180^\circ \), there is no such triangle by the Sum of the Angles of a Triangle Theorem.

There is one triangle with the given properties.

\( a = 6.8 \), \( b = 7 \), \( c \approx 2.05 \), \( \alpha \approx 76^\circ \), \( \beta = 87^\circ \), \( \gamma \approx 17^\circ \).

**Example.** Let \( \triangle ABC \) be the triangle with standard notations. Suppose \( \beta = 40^\circ \), \( a = 6.8 \) units, \( c = 7 \) units. Solve the triangle, if possible.

We cannot solve this triangle by using only the known theorems.

**Example.** Let \( \triangle ABC \) be the triangle with standard notations. Suppose \( a = 6.8 \), \( b = 5.7 \) units, \( c = 4.9 \) units. Solve the triangle, if possible.

We cannot solve this triangle by using only the known theorems. We need a new tool to solve the triangles in the last two examples.
9.4 Law of Cosines

We encountered two situations where we could not solve a given triangle by using the known theorems. We will develop a new tool for this purpose.

**Law of Cosines Theorem.** Consider a triangle ABC with standard notations. Then

1. \(a^2 = b^2 + c^2 - 2bc \cos \alpha\)
2. \(b^2 = c^2 + a^2 - 2ca \cos \beta\)
3. \(c^2 = a^2 + b^2 - 2ab \cos \gamma\)

**Proof.** We will prove \(a^2 = b^2 + c^2 - 2bc \cos \alpha\). The proofs of the other two parts are similar and therefore, left as exercises.

Drop a perpendicular from A to BC. Let the foot of this perpendicular be D. As we have observed before, D can lie within the segment BC or outside of the segment BC. Consider the case where D lies within BC.

By using the Pythagorean Theorem on the right triangle ACD we get:

\[|AC|^2 = |AD|^2 + |DC|^2\]

In this case, \(|DC| = |BC| - |BD|\). That is, \(|DC| = a - c \cos \beta\). Also, \(|AD| = c \sin \beta\). Therefore,

\[b^2 = (a - c \cos \beta)^2 + c^2 \sin^2 \beta\]

That is,

\[b^2 = a^2 - 2ac \cos \beta + c^2 \cos^2 \beta + c^2 \sin^2 \beta. \quad (1)\]
By using the Pythagorean Theorem on the right triangle $ABD$ we get:

$$|AB|^2 = |AD|^2 + |BD|^2$$

That is,

$$c^2 = c^2 \sin^2 \beta + c^2 \cos^2 \beta. \quad (2)$$

By substituting $c^2$ for $c^2 \sin^2 \beta + c^2 \cos^2 \beta$ in (1):

$$b^2 = a^2 - 2ac \cos \beta + c^2.$$  

or

$$b^2 = c^2 + a^2 - 2ca \cos \beta.$$  

Now consider the second case, where $D$ lies outside of $BC$.

By using the Pythagorean Theorem on the right triangle $ACD$ we get:

$$|AC|^2 = |AD|^2 + |CD|^2$$

In this case, $|CD| = |BD| - |BC|$. That is, $|CD| = c \cos \beta - a$. Also, $|AD| = c \sin \beta$. Therefore,

$$b^2 = (c \cos \beta - a)^2 + c^2 \sin^2 \beta.$$  

That is,

$$b^2 = c^2 \cos^2 \beta - 2ac \cos \beta + a^2 + c^2 \sin^2 \beta. \quad (3)$$

By using the Pythagorean Theorem on the right triangle $ABD$ we get:

$$|AB|^2 = |AD|^2 + |BD|^2$$
That is,
\[ c^2 = c^2 \sin^2 \beta + c^2 \cos^2 \beta. \] (4)

By substituting \( c^2 \) for \( c^2 \sin^2 \beta + c^2 \cos^2 \beta \) in (3):
\[ b^2 = c^2 - 2ac \cos \beta + a^2. \]

or
\[ b^2 = c^2 + a^2 - 2ca \cos \beta. \]

\[ \Box \]

**Exercise.** Consider a triangle \( ABC \) with standard notations. Show that
1. \( a^2 = b^2 + c^2 - 2bc \cos \alpha \)
2. \( c^2 = a^2 + b^2 - 2ab \cos \gamma \)

The Law of Cosines Theorem is a generalization of the Pythagorean Theorem in the following sense. Consider the statement \( c^2 = a^2 + b^2 - 2ab \cos \gamma \). Since \( \gamma \) is an angle of a triangle, the degree measure of \( \gamma \) can be any value between \( 0^\circ \) and \( 180^\circ \), exclusively. In the special case if \( \gamma = 90^\circ \), then \( \cos \gamma = 0 \) and we get, \( c^2 = a^2 + b^2 \).

**Example.** Let \( ABC \) be the triangle with standard notations. Suppose \( \beta = 40^\circ \), \( a = 6.8 \) units, \( c = 7 \) units. Solve the triangle, if possible.

**Answer.** By the SAS theorem, there is only one such triangle up to a congruence.

\[ \begin{align*}
\text{By the Law of Cosines Theorem,} \\
b^2 &= 7^2 + 6.8^2 - 2(7)(6.8) \cos 40^\circ
\end{align*} \]
Since $b > 0$,
\[ b = \sqrt{7^2 + 6.8^2 - 2(7)(6.8)\cos 40^\circ} \approx 4.7. \]

By the Law of Cosines Theorem,
\[ 6.8^2 = 7^2 + b^2 - 2(7)(b)\cos \alpha \]

That is,
\[ \cos \alpha \approx \frac{7^2 + 4.7^2 - 6.8^2}{2(7)(4.7)} \]

We want to find the solution to this equation between $0^\circ$ and $180^\circ$. By the definition of the inverse cosine function, the only solution of this equation between $0^\circ$ and $180^\circ$ is
\[ \alpha = \cos^{-1} \left( \frac{7^2 + 4.7^2 - 6.8^2}{2(7)(4.7)} \right) \approx 67.8^\circ. \]

Now, by the Sum of the Angles of a Triangle Theorem,
\[ \gamma = 180^\circ - \alpha - \beta \approx 72.2^\circ. \]

Therefore,
\[ a = 6.8 \text{ units}, \ b \approx 4.7 \text{ units}, \ c = 7 \text{ units}, \ \alpha \approx 67.8^\circ, \ \beta = 40^\circ, \ \text{and} \ \gamma \approx 72.2^\circ. \]

**Example.** Let $ABC$ be the triangle with standard notations. Suppose $a = 6.8$, $b = 5.7$ units, $c = 4.9$ units. Solve the triangle, if possible.

**Answer.** By the SSS theorem, there is only one such triangle up to a congruence.

\[ \begin{align*}
A \\
\quad 4.9 \\
\quad \alpha \\
\quad 5.7 \\
B \\
\quad \beta \\
\quad 6.8 \\
\quad \gamma \\
C \\
\end{align*} \]

By the Law of Cosines Theorem,
\[ 6.8^2 = 4.9^2 + 5.7^2 - 2(4.9)(5.7)\cos \alpha \]

That is,
\[ \cos \alpha \approx \frac{4.9^2 + 5.7^2 - 6.8^2}{2(4.9)(5.7)} \]
By the definition of the inverse cosine function, the only solution of this equation between 0° and 180° is
\[
\alpha = \cos^{-1}\left(\frac{4.9^2 + 5.7^2 - 6.8^2}{2(4.9)(5.7)}\right) \approx 79.4°.
\]

By the Law of Cosines Theorem,
\[
5.7^2 = 4.9^2 + 6.8^2 - 2(4.9)(6.8)\cos\beta
\]
That is,
\[
\cos\beta \approx \frac{4.9^2 + 6.8^2 - 5.7^2}{2(4.9)(6.8)}
\]
By the definition of the inverse cosine function, the only solution of this equation between 0° and 180° is
\[
\beta = \cos^{-1}\left(\frac{4.9^2 + 6.8^2 - 5.7^2}{2(4.9)(6.8)}\right) \approx 55.5°.
\]
Now, by the Sum of the Angles of a Triangle Theorem,
\[
\gamma = 180° - \alpha - \beta \approx 45.1°.
\]
Therefore,  
\[
a = 6.8\text{ units}, \ b = 5.7\text{ units}, \ c = 4.9\text{ units}, \ \alpha \approx 79.4°, \ \beta \approx 55.5°, \text{ and } \gamma \approx 45.1°. \]
Consider a given triangle with standard notations. Our next goal is to find the area of such a triangle. You may have learned in high school a theorem stating that the area of a triangle is “\( \frac{1}{2}(\text{base})(\text{height}) \)”. We will use this theorem and the knowledge of Trigonometry to find a few results that can be used to calculate the area of a triangle.

**Area of a Triangle Theorem.** Let \( \triangle ABC \) be a triangle with standard notations. Then

1. the area of the triangle \( = \frac{1}{2}bc \sin \alpha \)
2. the area of the triangle \( = \frac{1}{2}ca \sin \beta \)
3. the area of the triangle \( = \frac{1}{2}ab \sin \gamma \)

*Proof.* We will prove the second statement and leave the proof the other two as exercises. Drop a perpendicular from \( A \) to \( BC \). Let the foot of this perpendicular be \( D \). There are two cases; \( D \) lies within the segment \( BC \) or \( D \) lies outside of the segment \( BC \). Consider the case where \( D \) lies within the segment \( BC \) (In case \( \triangle ABC \) is a right angle with \( \gamma = 90^\circ \), then \( D = C \)).
Then $|AD| = c \sin \beta$. Therefore, the area of the triangle = $\frac{1}{2} \text{(base) (height)} = \frac{1}{2}(|BC|)(|AD|) = \frac{1}{2}a(c \sin \beta) = \frac{1}{2}ca \sin \beta$.

Now consider the case where $D$ lies outside of the segment $BC$.

Then $|AD| = c \sin \beta$. Therefore, the area of the triangle = $\frac{1}{2} \text{(base) (height)} = \frac{1}{2}(|BC|)(|AD|) = \frac{1}{2}a(c \sin \beta) = \frac{1}{2}ca \sin \beta$.

Exercise. Prove the other two parts of the theorem.

Example. Let $ABC$ be a triangle with standard notations and let $c = 7$ units, $\beta = 20^\circ$, and $a = 8$ units. Find the area of the triangle.

Answer. By the SAS Theorem, there is only one such triangle up to a congruence.

By the Area of a Triangle Theorem, the area of the triangle is $\frac{1}{2}(8)(7 \sin 20^\circ) \approx 9.58$ area units.

Example. Let $ABC$ be a triangle with standard notations and let $\beta = 20^\circ, a = 8$ units, and $\gamma = 40^\circ$. Find the area of the triangle.

Answer. By the ASA Theorem, there is only one such triangle up to a congruence. By the Sum of the Angles of a Triangle Theorem, $\alpha = 120^\circ$. 

By the Law of Sines Theorem,

\[ \frac{c}{\sin 40^\circ} = \frac{8}{\sin 120^\circ} \]

By the Cross Multiplication Algorithm Theorem,

\[ c = \frac{8 \sin 40^\circ}{\sin 120^\circ} \]

Now by the Area of a Triangle Theorem, the area of the triangle is

\[ \frac{1}{2} (8) \left( \frac{8 \sin 40^\circ}{\sin 120^\circ} \right) \sin 20^\circ \approx 8.12 \text{ area units}. \]

Example. Let \( ABC \) be a triangle with standard notations and let \( a = 8 \) units, \( b = 4 \) units, and \( c = 7 \) units. Find the area of the triangle.

Answer. By the SSS Theorem, there is only one such triangle up to a congruence.

By the Law of Cosines Theorem,

\[ 4^2 = 7^2 + 8^2 - 2(7)(8) \cos \beta \]

That is

\[ \cos \beta = \frac{7^2 + 8^2 - 4^2}{2(7)(8)} \]

and

\[ \beta = \cos^{-1} \left( \frac{7^2 + 8^2 - 4^2}{2(7)(8)} \right). \]

Now by the Area of a Triangle Theorem, the area of the triangle is

\[ \frac{1}{2} (7)(8) \sin \left( \cos^{-1} \left( \frac{7^2 + 8^2 - 4^2}{2(7)(8)} \right) \right) \approx 14 \text{ area units}. \]
10.1 Heron’s Formula

There is an ingenious theorem discovered by the Greek mathematicians about 2000 years ago to calculate the area of a triangle when you know the lengths of its three sides. Ancient Greeks visualized numbers as lengths of segments. That is, a product of two numbers represented an area, and the product of three numbers represented a volume. This theorem contains a product of four numbers, and it did not represent anything that they knew of. Yet they discovered that the square-root of a product of a certain four numbers associated with a given triangle is the area of the triangle. This theorem is known today as Heron’s Formula in honor of Heron of Alexandria (10 - 70 AD.)

Heron’s Formula Theorem. Let \( ABC \) be a triangle with standard notations. Let \( s \) be the semi-perimeter of the triangle. That is, \( s = \frac{(a + b + c)}{2} \). Then the area of the triangle is \( \sqrt{s(s - a)(s - b)(s - c)} \).

Let us re-do the previous example now using the Heron’s Formula Theorem before proving the Heron’s Formula Theorem.

Example. Let \( ABC \) be a triangle with standard notations and let \( a = 8 \) units, \( b = 4 \) units, and \( c = 7 \) units. Find the area of the triangle.

Answer. The semi-perimeter \( s \) of this triangle is \( \frac{8+4+7}{2} = \frac{19}{2} \). By the Heron’s Formula Theorem, the area of the triangle is

\[
\sqrt{\frac{19}{2} \left( \frac{19}{2} - 8 \right) \left( \frac{19}{2} - 4 \right) \left( \frac{19}{2} - 7 \right)} \approx 14 \text{ area units.}
\]

The following is the proof of the Heron’s Formula Theorem based on what we have learned up to now.

Proof. Let us examine what those four numbers are in the formula first. We know what \( s \) is.

\[
s - a = \frac{a + b + c}{2} - a = \frac{a + b + c}{2} - \frac{2a}{2} = \frac{b + c - a}{2}.
\]

\[
s - b = \frac{a + b + c}{2} - b = \frac{a + b + c}{2} - \frac{2b}{2} = \frac{a + c - b}{2}.
\]

(10.1)
10.1. HERON’S FORMULA

\[ s - c = \frac{a + b + c}{2} - c = \frac{a + b + c - 2c}{2} = \frac{a + b - c}{2}. \]

Let \( A \) be the area of the triangle. Then by the Area of a Triangle Theorem,

\[ A = \frac{1}{2}ca \sin \beta. \]

That is,

\[ A^2 = \frac{1}{4}c^2a^2 \sin^2 \beta. \]

By using the Pythagorean Theorem on the triangle \( ABD \) in either of the following figures, we get:

\[ c^2 = c^2 \cos^2 \beta + c^2 \sin^2 \beta. \]

The above equation can be written as:

\[ c^2 \sin^2 \beta = c^2 - c^2 \cos^2 \beta. \]

By multiplying both sides by \( a^2 \), we get:

\[ c^2a^2 \sin^2 \beta = c^2a^2 - c^2a^2 \cos^2 \beta. \]

Now substitute the above in \( A^2 \):

\[ A^2 = \frac{1}{4}(c^2a^2 - c^2a^2 \cos^2 \beta). \]

By the Law of Cosines Theorem,

\[ b^2 = c^2 + a^2 - 2ca \cos \beta. \]

The above equation can be written as:

\[ ca \cos \beta = \frac{1}{2}(c^2 + a^2 - b^2). \]
Substitute the above in $A^2$:

$$A^2 = \frac{1}{4} \left( c^2a^2 - \frac{1}{4}(c^2 + a^2 - b^2)^2 \right)$$

The above equation can be written as:

$$A^2 = \frac{1}{16} \left( 4c^2a^2 - (c^2 + a^2 - b^2)^2 \right) \quad (10.2)$$

By using the Difference of Squares Identity in (2):

$$A^2 = \frac{1}{16} \left( 2ca - (c^2 + a^2 - b^2) \right) \left( 2ca + (c^2 + a^2 - b^2) \right)$$

The above equation is the same as:

$$A^2 = \frac{1}{16} \left( b^2 - (c^2 - 2ca + a^2) \right) \left( (c^2 + 2ac + a^2) - b^2 \right)$$

By the Binomial Square Identities:

$$A^2 = \frac{1}{16} \left( b^2 - (c - a)^2 \right) \left( (c + a)^2 - b^2 \right) \quad (10.3)$$

By using the Difference of Squares Identity in (3):

$$A^2 = \frac{1}{16} (b - (c - a))(b + (c - a))(c + a - b)(c + a + b)$$

That is,

$$A^2 = \frac{1}{16} (a + b - c)(b + c - a)(c + a - b)(a + b + c)$$

The above equation is the same as:

$$A^2 = \frac{(a + b - c)}{2} \frac{(b + c - a)}{2} \frac{(c + a - b)}{2} \frac{(a + b + c)}{2}$$

By using (1):

$$A^2 = (s - c)(s - a)(s - b)s$$

Since $A > 0$,

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

Now to tie up some loose ends, notice that $a + b > c$, $b + c > a$ and $c + a > b$ by the Triangle Inequality Theorem. Therefore, $s - a$, $s - b$ and $s - c$ are all positive quantities.
Chapter 11

Applications: Trigonometric Identities

Recall that an algebraic equation that is true for all real numbers is called an identity. For example, $x^2 - a^2 = (x - a)(x + a)$ is true for any real number $x$ and for any real number $a$. We relaxed the strict requirement that an identity must be true for all real numbers to include the following as an algebraic identity.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \text{ for all real numbers } a, b, c, \text{ and } d \text{ except } b = 0 \text{ and } c = 0.$$ 

The above identity is true for all real numbers except for few real numbers.

We will relax the requirement even further to include an important list of trigonometric equations as trigonometric identities. For example, we want to call the following a trigonometric identity.

$$\tan x = \frac{\sin x}{\cos x}, \text{ where } x \text{ is a real number}.$$ 

As you know that there are countably many real numbers where $\cos x = 0$. The general solution of the equation $\cos x = 0$ is $2n\pi \pm \frac{\pi}{2}$, for any integer $n$. In other words, the above equation is not true for countably many numbers. However, the equation is true for many more numbers.\(^1\)

\(^1\)The equation is true for uncountably many numbers. In mathematical parlance, the identity is true for all numbers except for a set whose measure is 0. The explanation of what that means is beyond our comprehension at the moment.
11.1 Basic Trigonometric identities

We will start with definitions of tangent, secant, cosecant, and cotangent numbers of a given real number. They are trigonometric identities by the above description of a trigonometric identity.

Let $\theta$ be any real number. Then

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (A)
\]
\[
\csc \theta = \frac{1}{\sin \theta}
\]
\[
\sec \theta = \frac{1}{\cos \theta}
\]
\[
\cot \theta = \frac{\cos \theta}{\sin \theta}
\]

Let us revisit the unit circle to identify several more basic trigonometric identities.\(^2\) For co-terminal angles in standard position, the following are always true.

Suppose $\theta$ is a given angle in radians. Then there is an angle $\alpha$ in the standard position so that $0 \leq \alpha < 2\pi$ and $\theta$ and $\alpha$ are co-terminal. That is, there is an integer $k$ so that $\theta = 2k\pi + \alpha$.

\(^2\)You can obtain the same identities by examining the graphs of trigonometric functions.
Since $\theta$ and $\alpha$ are co-terminal angles, the point of intersection of the terminal side and the unit circle, say $P(a, b)$, is the same for both angles. In addition, any angle of the form $2n\pi + \theta$ is co-terminal with $\theta$, for any integer $n$. Therefore, we have the following two identities.

For any angle $\theta$ and for any integer $n$,

\[
\begin{align*}
\sin(2n\pi + \theta) &= \sin \theta \\
\cos(2n\pi + \theta) &= \cos \theta
\end{align*}
\]

Since any given angle $\theta$ in standard position is co-terminal to an angle $\alpha$ in standard position, where $0 \leq \alpha < 2\pi$, and $\sin \theta = \sin \alpha$ and $\cos \theta = \cos \alpha$, we will just look at an arbitrary angle in $[0, 2\pi)$ to obtain the following identities.

Consider any angle $\theta$ in the standard position. Let $P(a, b)$ be the point of intersection between the terminal side of $\theta$ and the unit circle. We can obtain the angle $-\theta$ in the standard position by reflecting the angle $\theta$ about the $x$-axis. Then the point of intersection $Q$ between the terminal side of $-\theta$ and the unit circle has coordinates $(a, -b)$.

As a result we have two new identities.
For any angle \( \theta \),
\[
\begin{align*}
\cos(-\theta) &= \cos \theta \\
\sin(-\theta) &= -\sin \theta
\end{align*}
\]  
(C)

Consider any angle \( \theta \) in the standard position. Let \( P(a, b) \) be the point of intersection of \( \theta \) and the unit circle. Then \( a = \cos \theta \) and \( b = \sin \theta \), by definition.

By the Distance Formula Theorem,
\[
|PO| = \sqrt{(\cos \theta - 0)^2 + (\sin \theta - 0)^2}
\]

But \(|PO|\) is the radius of the unit circle. That is, \(|PO| = 1\). Therefore,
\[
1 = \sqrt{(\cos \theta)^2 + (\sin \theta)^2}
\]

By squaring both sides of the above equation we get the following important trigonometric identity.

For any angle \( \theta \),
\[
\cos^2 \theta + \sin^2 \theta = 1.
\]  
(D)
By subtracting $\sin^2 \theta$ from both sides of (D), you get the following trigonometric identity.
For any angle $\theta$,
\[
\cos^2 \theta = 1 - \sin^2 \theta.
\]

By subtracting $\cos^2 \theta$ from both sides of (D), you get the following trigonometric identity.
For any angle $\theta$,
\[
\sin^2 \theta = 1 - \cos^2 \theta.
\]

By dividing both sides of (D) by $\cos^2 \theta$ and using the identities in (A), you get the following trigonometric identity.
For any angle $\theta$,
\[
1 + \tan^2 \theta = \sec^2 \theta.
\]

By subtracting 1 from both sides of the above identity, you get the following identity.
For any angle $\theta$,
\[
\tan^2 \theta = \sec^2 \theta - 1.
\]

By dividing both sides of (D) by $\sin^2 \theta$ and using the identities in (A), you get the following trigonometric identity.
For any angle $\theta$,
\[
\cot^2 \theta + 1 = \csc^2 \theta.
\]

By subtracting 1 from both sides of the above identity, you get the following identity.
For any angle $\theta$,
\[
\cot^2 \theta = \csc^2 \theta - 1.
\]

Since all the previous six identities are direct descendants of the identity (D) we will list them together.
For any angle \( \theta \),
\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\cos^2 \theta &= 1 - \sin^2 \theta \\
\sin^2 \theta &= 1 - \cos^2 \theta \\
1 + \tan^2 \theta &= \sec^2 \theta \\
\tan^2 \theta &= \sec^2 \theta - 1 \\
\cot^2 \theta + 1 &= \csc^2 \theta \\
\cot^2 \theta &= \csc^2 \theta - 1
\end{align*}
\] (E)

We need one more basic trigonometric identity before we proceed. For this we need to use another theorem that you may have learned in high school.

**Slopes of Perpendicular Lines Theorem.** Suppose two lines \( \ell_1 \) and \( \ell_2 \) are perpendicular to each other. Suppose the slope of the line \( \ell_1 \) is \( m_1 \) and the slope of the line \( \ell_2 \) is \( m_2 \). Then \( m_1 \cdot m_2 = -1 \).

Let us first observe a few properties of the coordinate system. Let us pick a point \( P_1(a_1, b_1) \) in the first quadrant. Then the signs of the coordinates of \( P_1 \) are both positive. We will indicate this by \((+,+)\). Rotate the plane about \( O \) by an angle of \( \frac{\pi}{2} \) counterclockwise. Suppose the rotated \( P_1 \) is \( P_2(a_2, b_2) \). Then \( P_2 \) is in the second quadrant and the signs of coordinates of \( P_2 \) are \((-,+\)) . Rotate the plane about \( O \) by an angle of \( \frac{\pi}{2} \) counterclockwise again. Suppose the rotated \( P_2 \) is \( P_3(a_3, b_3) \). Then \( P_3 \) is in the third quadrant and the signs of coordinates of \( P_3 \) are \((-,-)\). Rotate the plane about \( O \) by an angle of \( \frac{\pi}{2} \) counterclockwise again. Suppose the rotated \( P_3 \) is \( P_4(a_4, b_4) \). Then \( P_4 \) is in the fourth quadrant and the signs of coordinates of \( P_4 \) are \((+,-)\).
If we rotate the plane one more time about $O$ by an angle of $\frac{\pi}{2}$ counterclockwise, then the image of $P_4$ under this rotation will be $P_1$.

Notice that after each rotation, the sign of the $y$-coordinate of the new point is the sign of the $x$-coordinate of the previous point. For example, the sign of the $y$-coordinate of $P_3$ is the sign of the $x$-coordinate of $P_2$. Also notice that the sign of the $x$-coordinate of the new point is the opposite sign of the $y$-coordinate of the previous point.

Next we will look at the relationships of the coordinates of a point and the rotated image of the point. Without loss of generality, let us look at $P_1$ and $P_2$. Let us assume that $P_1$ lies on the unit circle. Then $P_2$ lies on the unit circle as well, as rotations preserve lengths of line segments. The slope of the line $L_{OP_1}$ is $\frac{b_1 - 0}{a_1 - 0} = \frac{b_1}{a_1}$. The slope of the line $L_{OP_2}$ is $\frac{b_2 - 0}{a_2 - 0} = \frac{b_2}{a_2}$. Since $L_{OP_1}$ is perpendicular to $L_{OP_2}$, by the Slopes of the Perpendicular Lines Theorem,

$$\left(\frac{b_1}{a_1}\right) \cdot \left(\frac{b_2}{a_2}\right) = -1.$$

Therefore, by the Cross-Multiplication Algorithm Theorem,

$$a_2 = -\left(\frac{b_1}{a_1}\right) b_2.$$

Since, $P_2$ is a point on the unit circle,

$$a_2^2 + b_2^2 = 1.$$

$$\Rightarrow \left(-\left(\frac{b_1}{a_1}\right) b_2\right)^2 + b_2^2 = 1.$$
\[ b_2^2 \left( \left( \frac{b_1}{a_1} \right)^2 + 1 \right) = 1. \]
\[ b_2^2 \left( \frac{b_1^2 + a_1^2}{a_1^2} \right) = 1. \]

Since \((a_1, b_1)\) is on the unit circle, \(a_1^2 + b_1^2 = 1\). Therefore,
\[ b_2^2 \left( \frac{1}{a_1^2} \right) = 1. \]

By the Cross Multiplication Algorithm Theorem,
\[ b_2^2 = a_1^2. \]

By the Square-Root Principle Theorem,
\[ b_2 = \pm a_1. \]

By substituting the values of \(b_2\) into \(a_2\),
\[ a_2 = \mp b_1. \]

In other words, \((a_2, b_2)\) is either \((b_1, -a_1)\) or \((-b_1, a_1)\). Since the rotation is counterclockwise, \((a_2, b_2)\) is \((-b_1, a_1)\). As a result, we have the following identities.

For any angle \(\theta\),
\[
\begin{align*}
\cos \left( \theta + \frac{\pi}{2} \right) &= -\sin \theta \\
\sin \left( \theta + \frac{\pi}{2} \right) &= \cos \theta
\end{align*}
\]

We claim that the identities in \((H)\) are true for any angle \(\theta\). However we did not show that the above identities are true if the terminal side of the angle \(\theta\) is on one of the axes. I will leave it as an exercise for you to prove that fact.

**Exercise.** Show that the identities in \((H)\) are true if the terminal side of \(\theta\) lies on one of the axes.
11.2 Sum and Difference Identities

The next four trigonometric identities that we are going to prove are extremely important. I would like to refer to these four trigonometric identities as the super-stars of trigonometric identities.

Consider two angles $\alpha$ and $\beta$. Without loss of generality, let us assume that $\alpha \geq \beta$. Since any angle is a co-terminal angle of an angle in the interval $[0, 2\pi)$, we will also assume that both $\alpha$ and $\beta$ are in the interval $[0, 2\pi)$. Let $P$ be the point of intersection of the terminal side of $\alpha$ and the unit circle, and let $Q$ be the point of intersection of the terminal side of $\beta$ and the unit circle. Then $P$ has the coordinates $(\cos \alpha, \sin \alpha)$ and $Q$ has the coordinates $(\cos \beta, \sin \beta)$.

By the Distance Formula Theorem, the length of the segment $PQ$ is:

$$|PQ| = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2}.$$

Now rotate the plane about $O$ by an angle $\beta$ clockwise. Suppose the image of $P$ under this rotation is $P'$ and the image of $Q$ under this rotation is $Q'$. 

By the Distance Formula Theorem, the length of the segment $PQ$ is:
Since rotations preserve the measures of angles, the angle $\angle Q'OP' = \alpha - \beta$. Since the angle $\angle Q'OP'$ is in the standard position, the coordinates of $P'$ are $(\cos(\alpha - \beta), \sin(\alpha - \beta))$ and the coordinates of $Q'$ are $(1, 0)$. Therefore, the length of the segment $P'Q'$ is:

$$|P'Q'| = \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2}.$$ 

Since rotations preserve lengths of segments, $|PQ| = |P'Q'|$. Therefore,

$$\sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} = \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta))^2}.$$ 

By squaring both sides, we get:

$$(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = (\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta))^2.$$ 

By using the Binomial Identity:

$$(\cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta)$$

$$= (\cos^2(\alpha - \beta) - 2 \cos(\alpha - \beta) + 1) + \sin^2(\alpha - \beta).$$

By using the identity $(D)$, we get:

$$(1 - 2 \cos \alpha \cos \beta) + (1 - 2 \sin \alpha \sin \beta) = (2 - 2 \cos(\alpha - \beta)).$$
Subtract 2 from both sides:

\[ 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta = -2 \cos(\alpha - \beta). \]

By dividing both sides by \(-2\) we get:

For any angle \(\alpha\) and for any angle \(\beta\),

\[ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta. \quad (I) \]

For any angle \(\alpha\) and for any angle \(\beta\), \(\alpha + \beta = \alpha - (-\beta)\). Therefore, by using the identity \((I)\) we get:

\[ \cos(\alpha + \beta) = \cos(\alpha - (-\beta)) \]
\[ = \cos \alpha \cos(-\beta) + \sin \alpha \sin(-\beta). \]

By the identity \((C)\), \(\cos(-\beta) = \cos \beta\) and \(\sin(-\beta) = -\sin \beta\). Therefore:

For any angle \(\alpha\) and for any angle \(\beta\),

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \quad (J) \]

By the identity \((H)\), \(\sin(\alpha + \beta) = -\cos \left( \frac{\pi}{2} + (\alpha + \beta) \right)\). Therefore,

\[ \sin(\alpha + \beta) = -\cos \left( \frac{\pi}{2} + (\alpha + \beta) \right) \]
\[ = -\cos \left( \left( \frac{\pi}{2} + \alpha \right) + \beta \right) \]
\[ = -\left( \cos \left( \frac{\pi}{2} + \alpha \right) \cos \beta - \sin \left( \frac{\pi}{2} + \alpha \right) \sin \beta \right) \]
\[ = -\cos \left( \frac{\pi}{2} + \alpha \right) \cos \beta + \sin \left( \frac{\pi}{2} + \alpha \right) \sin \beta \]

by the identity \((J)\). Now by using the identity \((H)\) again, we get the following identity.

For any angle \(\alpha\) and for any angle \(\beta\),

\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (K) \]
For any angle $\alpha$ and for any angle $\beta$, $\alpha - \beta = \alpha + (-\beta)$. Therefore, by using identity (K), we get:

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta).$$

By identity (C), $\cos(-\beta) = \cos \beta$ and $\sin(-\beta) = -\sin \beta$. Therefore:

$$\begin{align*}
\text{For any angle } \alpha \text{ and for any angle } \beta, \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta. \\
\text{(L)}
\end{align*}$$

The identities (I) – (L) are known as the Sum-Difference Identities. These are the superstars of all trigonometric identities. Now let us demonstrate some of the powers of the identities (I) – (L).

**Example.** If we choose $\alpha = \beta$ in identity (I) we get

$$\cos(\alpha - \alpha) = \cos \alpha \cos \alpha + \sin \alpha \sin \alpha$$

That is,

$$\cos 0 = \cos^2 \alpha + \sin^2 \alpha$$

Since $\cos 0 = 1$, we get identity (D).

$$1 = \cos^2 \alpha + \sin^2 \alpha. \quad \square$$

**Example.** If we choose $\alpha = 0$ in identity (I) we get

$$\cos(0 - \beta) = \cos 0 \cos \beta + \sin 0 \sin \beta$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, we get one of the identities in (B).

$$\cos(-\beta) = \cos \beta. \quad \square$$

**Example.** If we choose $\alpha = 0$ in identity (L), we get

$$\sin(0 - \beta) = \sin 0 \cos \beta - \cos 0 \sin \beta$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, we get the identity in (B).

$$\sin(-\beta) = -\sin \beta. \quad \square$$
11.2. SUM AND DIFFERENCE IDENTITIES

Since we know the trigonometric numbers of special angles, by using those values with our super-star identities, we can find exact trigonometric numbers of a few more angles.

Example.

\[
\sin(15^\circ) = \sin(45^\circ - 30^\circ) \\
= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\
= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{1}}{2} \\
= \frac{\sqrt{3} - 1}{2\sqrt{2}}.
\]

Exercise.

1. Find the exact value of \(\cos(15^\circ)\).
2. Find the exact value of \(\cos(75^\circ)\).
3. Find the exact value of \(\sin(75^\circ)\).
4. Find the exact value of \(\cos(105^\circ)\).
5. Find the exact value of \(\sin(105^\circ)\).

Exercise. Show that for any angle \(\alpha\) and for any angle \(\beta\), where \(\tan \alpha\) and \(\tan \beta\) is defined,

\[
\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.
\] (M)

Exercise. Show that for any angle \(\alpha\) and for any angle \(\beta\), where \(\tan \alpha\) and \(\tan \beta\) is defined,

\[
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}.
\] (N)

In the next two sections, we will see many more applications of the super-star identities.
11.3 Double-angle identities

The following are the super-star identities.

For any angle $\alpha$ and for any angle $\beta$,

\begin{align*}
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta
\end{align*}

If we choose $\beta = \alpha$ in (II), then we get:

\[ \cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \]

That is,

For any angle $\alpha$

\[ \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \]

The identity in (O) is known as one of the double-angle identities. The identities (D) and (O) together form a nice pair of identities.\(^3\)

For any angle $\alpha$

\[ 1 = \cos^2 \alpha + \sin^2 \alpha \]

\[ \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \]

\(^3\)There is a class of functions called the Hyperbolic Functions. You will learn about them in an advanced algebra course or in a calculus course. There is a pair of hyperbolic identities similar to these two trigonometric identities. Namely $1 = \cosh^2 \alpha - \sinh^2 \alpha$ and $\cosh 2\alpha = \cosh^2 \alpha + \sinh^2 \alpha$. 
By using the identity $\cos^2 \alpha = 1 - \sin^2 \alpha$ in identity (O), we get:

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

By using the identity $\sin^2 \alpha = 1 - \cos^2 \alpha$ in identity (O), we get:

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

Since the last two identities are direct descendants of the identity (O), we will list them together.

For any angle $\alpha$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \quad (P)$$
$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$
$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

If we choose $\beta = \alpha$ in identity $K$, we get another double-angle identity.

$$\sin(\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

That is,

For any angle $\alpha$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad (Q)$$

Identities listed as $(P)$ and $(Q)$ are called double-angle identities.

**Example.** Suppose $\cos \theta = \frac{2}{3}$ and $\sin \theta < 0$. Find the exact values of $\sin 2\theta$ and $\cos 2\theta$. Decide in what quadrant the terminal side lies when the angle $2\theta$ is in standard position.

**Answer.** Since $\cos \theta > 0$ and $\sin \theta < 0$, when $\theta$ is in the standard position, the terminal side is in the fourth quadrant. That is, $\frac{3\pi}{2} < \theta < 2\pi$. 
Let $P(2, b)$ be a point on the terminal side of the angle $\theta$, where $b$ is a constant. Then by the definition of $\cos \theta$, $|OP| = 5$. By the distance formula:

$$|OP| = \sqrt{(2 - 0)^2 + (b - 0)^2}$$

$$\implies 5 = \sqrt{2^2 + b^2}$$

$$\implies 25 = 4 + b^2$$

$$\implies b^2 = 21$$

$$\implies b = \pm \sqrt{21}$$

Since $P$ is a point in the fourth quadrant $b = -\sqrt{21}$. Therefore, $\sin \theta = -\frac{\sqrt{21}}{5}$. Now by using double-angle identities:

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$= 2 \left( -\frac{\sqrt{21}}{5} \right) \left( \frac{2}{5} \right)$$

$$= -\frac{4\sqrt{21}}{25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= \left( \frac{2}{5} \right)^2 - \left( -\frac{\sqrt{21}}{5} \right)^2$$

$$= \frac{4}{25} - \frac{21}{25}$$

$$= -\frac{17}{25}$$

Since both $\cos 2\theta$ and $\sin 2\theta$ are negative, when the angle $2\theta$ is in the standard position their terminal side lies in the third quadrant.
Exercise. Show that for any angle $\alpha$, where $\tan \alpha$ is defined,

$$\tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha}.$$  \hfill (R)

11.4 Half-Angle Identities

In the last section we have discovered identities for $\cos 2\alpha$ and $\sin 2\alpha$ in terms of $\cos \alpha$ and $\sin \alpha$, for any given angle $\alpha$. Similarly, we would like to obtain identities for $\cos \frac{\alpha}{2}$ and $\sin \frac{\alpha}{2}$ in terms of $\cos \alpha$ and $\sin \alpha$, for any given angle $\alpha$.

By using identity $(P)$, for any given angle $\alpha$ we get:

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\Rightarrow 2 \cos^2 \alpha = 1 + \cos 2\alpha$$

$$\Rightarrow \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

Let $\theta = 2\alpha$. Then $\alpha = \frac{\theta}{2}$. Then the previous identity is:

$$\cos^2 \left( \frac{\theta}{2} \right) = \frac{1 + \cos \theta}{2}.$$  \hfill (S)

By the Square-Root Principle Theorem,

For any angle $\theta$,

$$\cos \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$  \hfill (S)

By using identity $(P)$ again, for any angle $\alpha$ we get:

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

$$\Rightarrow 2 \sin^2 \alpha = 1 - \cos 2\alpha$$

$$\Rightarrow \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$
Let $\theta = 2\alpha$. Then $\alpha = \frac{\theta}{2}$. Then the previous identity is:

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{2}.$$ 

By the Square-Root Principle Theorem,

For any angle $\theta$,

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{(T)}$$

**Example.** Find $\cos\left(\frac{\theta}{2}\right)$ and $\sin\left(\frac{\theta}{2}\right)$, if $\sin \theta = \frac{3}{5}$ and $\cos \theta < 0$. Assume that $\theta$ is in standard position.

![Diagram](image)

**Answer.** Since $\sin \theta > 0$ and $\cos \theta < 0$, the terminal side of $\theta$ is in the second quadrant. Let $P$ be the point on the terminal side of $\theta$ so that $|OP| = 5$. Then by the definition of $\sin \theta$, the $y$-coordinate of $P$ is 3. Assume that the $x$-coordinate of $P$ is $a$. Then by the Distance Formula Theorem,

$$5 = \sqrt{a^2 + 3^2}$$

$$\Rightarrow \quad 25 = a^2 + 9$$

$$\Rightarrow \quad a^2 = 16$$

$$\Rightarrow \quad a = \pm 4$$

Since $P$ is in the second quadrant, $a = -4$. Therefore, $\cos \theta = -\frac{4}{5}$. 
By using identity \((S)\),
\[
\cos \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 + \cos \theta}{2}}
\]
\[
= \pm \sqrt{\frac{1 + \frac{4}{5}}{2}}
\]
\[
= \pm \sqrt{\frac{9}{10}}
\]

Since \(\frac{\pi}{2} < \theta < \pi\), that is, \(\theta\) is in the second quadrant, \(\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}\). Therefore, \(\frac{\theta}{2}\) is in the first quadrant and \(\cos \left( \frac{\theta}{2} \right) = \frac{3}{\sqrt{10}}\).

By using the identity \((T)\),
\[
\sin \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}
\]
\[
= \pm \sqrt{\frac{1 - \frac{4}{5}}{2}}
\]
\[
= \pm \sqrt{\frac{1}{10}}
\]

Since \(\frac{\theta}{2}\) is in the first quadrant, \(\sin \left( \frac{\theta}{2} \right) = \frac{1}{\sqrt{10}}\).

**Exercise.** Show that for any angle \(\theta\), where \(\cos \theta \neq -1\),
\[
\tan \left( \frac{\theta}{2} \right) = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}. \quad (U)
\]

### 11.5 Product-to-Sum Identities

Let us look at the super-star identities again.
For any angle $\alpha$ and for any angle $\beta$, 

\[
\begin{align*}
\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad \text{(I)} \\
\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \text{(J)} \\
\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad \text{(K)} \\
\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad \text{(L)}
\end{align*}
\]

If we subtract $(J)$ from $(I)$, we get,

\[
\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta
\]

That is,

\[
\begin{align*}
\sin \alpha \sin \beta &= -\frac{1}{2} \left[ \cos(\alpha + \beta) - \cos(\alpha - \beta) \right] \quad \text{(PTS 1)}
\end{align*}
\]

If we add $(I)$ and $(J)$, we get,

\[
\cos(\alpha - \beta) + \cos(\alpha + \beta) = 2 \cos \alpha \sin \beta
\]

That is,

\[
\begin{align*}
\cos \alpha \cos \beta &= \frac{1}{2} \left[ \cos(\alpha + \beta) + \cos(\alpha - \beta) \right] \quad \text{(PTS 2)}
\end{align*}
\]

If we subtract $(L)$ from $(K)$, we get,

\[
\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta
\]

That is,
For any angle $\alpha$ and for any angle $\beta$,
\[
\cos \alpha \sin \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) - \sin(\alpha - \beta) \right]
\] (PTS 3)

If we add $(K)$ and $(L)$, we get,
\[
\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2 \sin \alpha \cos \beta
\]
That is,

For any angle $\alpha$ and for any angle $\beta$,
\[
\sin \alpha \cos \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) + \sin(\alpha - \beta) \right]
\] (PTS 4)

The identities PTS 1 - 4 are known as the Product-to-Sum Identities.

For any angle $\alpha$ and for any angle $\beta$,
\[
\sin \alpha \sin \beta = -\frac{1}{2} \left[ \cos(\alpha + \beta) - \cos(\alpha - \beta) \right]
\] (PTS 1)
\[
\cos \alpha \cos \beta = \frac{1}{2} \left[ \cos(\alpha + \beta) + \cos(\alpha - \beta) \right]
\] (PTS 2)
\[
\cos \alpha \sin \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) - \sin(\alpha - \beta) \right]
\] (PTS 3)
\[
\sin \alpha \cos \beta = \frac{1}{2} \left[ \sin(\alpha + \beta) + \sin(\alpha - \beta) \right]
\] (PTS 4)

### 11.6 Sum-to-Product Identities

Let $A = \alpha + \beta$ and $B = \alpha - \beta$ in identities PST 1 - 4.

If we add $A$ and $B$, we get,
\[
2\alpha = A + B
\]
That is, $\alpha = \frac{A + B}{2}$.

If we subtract $B$ from $A$, we get,

$$2\beta = A - B$$

That is, $\beta = \frac{A - B}{2}$.

Now substitute $A$ for $\alpha + \beta$, $B$ for $\alpha - \beta$, $\frac{A+B}{2}$ for $\alpha$, and $\frac{A-B}{2}$ for $\beta$ in the identities PST 1 - 4. Then we get,

For any angle $A$ and for any angle $B$,

$$\sin \left( \frac{A+B}{2} \right) \sin \left( \frac{A-B}{2} \right) = -\frac{1}{2} \left[ \cos A - \cos B \right]$$

$$\cos \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right) = \frac{1}{2} \left[ \cos A + \cos B \right]$$

$$\cos \left( \frac{A+B}{2} \right) \sin \left( \frac{A-B}{2} \right) = \frac{1}{2} \left[ \sin A - \sin B \right]$$

$$\sin \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right) = \frac{1}{2} \left[ \sin A + \sin B \right]$$

Multiply each identity by 2 and we get the four identities known as the Sum-to-Product Identities.
For any angle \( A \) and for any angle \( B \),

\[
\begin{align*}
\cos A - \cos B &= 2 \sin \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right) \quad \text{(STP 1)} \\
\cos A + \cos B &= 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \quad \text{(STP 2)} \\
\sin A - \sin B &= 2 \cos \left( \frac{A + B}{2} \right) \sin \left( \frac{A - B}{2} \right) \quad \text{(STP 3)} \\
\sin A + \sin B &= 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \quad \text{(STP 4)}
\end{align*}
\]

If you can remember the identities PTS 1 - 4 and STP 1 - 4, then that is great. If you do not want to remember these identities, then you should be able to derive them from the four super-star identities as demonstrated. That is, if nothing else, you must remember the super-stars.

11.7 Other Trigonometric Identities and Applications

The trigonometric identities (A) - (U), PTS 1 - 4, and STP 1 - 4 are known as basic trigonometric identities. You can derive or prove the validity of any other trigonometric identity using the basic trigonometric identities, algebraic identities, along with some basic algebraic methods such as using the distributive property and collecting like terms. The reason why we can do this is ultimately related to the fact that every letter represents a real number and the trigonometric numbers of an angle are real numbers.

For example, we can prove the following trigonometric identity using the difference of squares identity and identities (D) and (M).

**Example.** Prove that, for any numbers \( A \)

\[
\cos^4 A - \sin^4 A = \cos 2A
\]

**Proof.**

Left Side  = \cos^4 A - \sin^4 A
\[= (\cos^2 A - \sin^2 A)(\cos^2 A + \sin^2 A), \text{ by the difference of squares identity.}\]
\[= (\cos 2A) \cdot (1), \text{ by identities (D) and (M).}\]
\[= \text{Right Side, for any number } A.\]

**Example.** Prove that, for any numbers \( A \) and \( B \),
\[
\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B
\]

**Proof.**
Left Side \( = \sin(A + B) \sin(A - B) \)
\[= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B), \]
by two of the super-stars
\[= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B, \text{ by the difference of squares identity.}\]

At this point of the proof we realize that there are no cosine terms on the right side. Therefore, we could replace \( \cos^2 A \) by \( 1 - \sin^2 A \), and \( \cos^2 B \) by \( 1 - \sin^2 B \). With this thought, let us go back to the proof of the identity.

Left Side \( = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \)
\[= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B \]
\[= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B, \text{ by the distributive property} \]
\[= \sin^2 A - \sin^2 B, \text{ by collecting like terms.}\]
\[= \text{Right Side.}\]

**Exercise.** Prove the identity in the above example by using the product-to-sum identities.

**Exercise.** Prove that \( \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta \).

[Hint: Write \( \cos 3\theta \) as \( \cos(2\theta + \theta) \) and use the super-star identities.]

**Exercise.** Prove that \( \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \).

**Example.** Find the exact value of \( \sin 18^\circ \).
Answer. 18° is $\frac{\pi}{10}$ in radians. Let $\theta = \frac{\pi}{10}$. Then $5\theta = \frac{\pi}{2}$. Since $\cos \frac{\pi}{2} = 0$, $\cos 5\theta = 0$.

Now we will deploy the super-stars.

$\cos 5\theta = \cos (3\theta + 2\theta)$

$= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta$

$= (4 \cos^3 \theta - 3 \cos \theta) \cos 2\theta - (3 \sin \theta - 4 \sin^3 \theta) \sin 2\theta$

$= (4 \cos^3 \theta - 3 \cos \theta)(\cos^2 \theta - \sin^2 \theta) - (3 \sin \theta - 4 \sin^3 \theta)(2 \sin \theta \cos \theta)$

$= 4 \cos^5 \theta - 4 \cos^3 \theta \sin^2 \theta - 3 \cos^3 \theta + 3 \cos \theta \sin^2 \theta - 6 \sin^2 \theta \cos \theta + 8 \sin^4 \theta \cos \theta$

$= \cos \theta(4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta - 3 \cos^2 \theta - 3 \sin^2 \theta + 8 \sin^4 \theta)$

$= \cos \theta(4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta - 3(\cos^2 \theta + \sin^2 \theta) + 8 \sin^4 \theta)$

$= \cos \theta(4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta + 8 \sin^4 \theta - 3)$

Since $\cos 5\theta = 0$, we get:

$\cos \theta(4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta + 8 \sin^4 \theta - 3) = 0$

Since $\theta = \frac{\pi}{10}$, $\cos \theta \neq 0$. Therefore,

$\cos \theta(4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta + 8 \sin^4 \theta - 3) = 0$

$\implies 4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta + 8 \sin^4 \theta - 3 = 0$

Since we are interested in finding $\sin \theta$, we will use $\cos^2 \theta = 1 - \sin^2 \theta$ to replace every $\cos^2 \theta$.

$4 \cos^4 \theta - 4 \cos^2 \theta \sin^2 \theta + 8 \sin^4 \theta - 3 = 0$

$\implies 4(1 - \sin^2 \theta)^2 - 4(1 - \sin^2 \theta) \sin^2 \theta + 8 \sin^4 \theta - 3 = 0$

$\implies 4 - 8 \sin^2 \theta + 4 \sin^2 \theta - 4 \sin^2 \theta + 4 \sin^4 \theta + 8 \sin^2 \theta - 3 = 0$

$\implies 16 \sin^4 \theta - 12 \sin^2 \theta + 1 = 0$

This is a quadratic equation in $\sin^2 \theta$. The discriminant is $12^2 - 4(16)(1) = (16)(5)$.

Therefore, by the quadratic formula,

$\sin^2 \theta = \frac{12 \pm \sqrt{5}}{32}$

$= \frac{3 \pm \sqrt{5}}{8}$
Therefore
\[
\sin \theta = \pm \sqrt{\frac{3 \pm \sqrt{5}}{8}}
\]

Since \( \theta \) is in the first quadrant,
\[
\sin \theta = \frac{1}{2} \sqrt{\frac{3 \pm \sqrt{5}}{2}}
\]

Clearly, \( \frac{3 + \sqrt{5}}{2} > 1 \). Therefore, \( \sqrt{\frac{3 + \sqrt{5}}{2}} > 1 \). Then, \( \frac{1}{2} \sqrt{\frac{3 + \sqrt{5}}{2}} > \frac{1}{2} = \sin \frac{\pi}{6} \). But this is impossible since \( \theta < \frac{\pi}{6} \).

Therefore,
\[
\sin \theta = \frac{1}{2} \sqrt{\frac{3 - \sqrt{5}}{2}}.
\]

Example. Show that
1. \( \cos \left( \frac{\pi}{2} - \alpha \right) = \sin \alpha \)
2. \( \sin \left( \frac{\pi}{2} - \alpha \right) = \cos \alpha \)

Answer. By one of the super-stars,
\[
\cos \left( \frac{\pi}{2} - \alpha \right) = \cos \frac{\pi}{2} \cos \alpha + \sin \frac{\pi}{2} \sin \alpha
\]
\[
= 0 \cdot \cos \alpha + 1 \cdot \sin \alpha
\]
\[
= \sin \alpha.
\]

By another super-star,
\[
\sin \left( \frac{\pi}{2} - \alpha \right) = \sin \frac{\pi}{2} \cos \alpha - \cos \frac{\pi}{2} \sin \alpha
\]
\[
= 1 \cdot \cos \alpha - 0 \cdot \sin \alpha
\]
\[
= \cos \alpha. \quad \square
\]

Exercise. Find the exact value of \( \sin 18^\circ \) by using the following method.

As before, let \( \theta = \frac{\pi}{10} \). Then \( 5\theta = \frac{\pi}{2} \) or \( 3\theta + 2\theta = \frac{\pi}{2} \). We can write this equation as
\[
3\theta = \frac{\pi}{2} - 2\theta
\]
Therefore,
\[
\cos 3\theta = \cos \left( \frac{\pi}{2} - 2\theta \right),
\]
\[\Rightarrow 4 \cos^3 \theta - 3 \cos \theta = \sin 2\theta\]
\[\Rightarrow 4 \cos^3 \theta - 3 \cos \theta - \sin 2\theta = 0\]
\[\Rightarrow 4 \cos^3 \theta - 3 \cos \theta - 2 \sin \theta \cos \theta = 0\]
\[\Rightarrow \cos \theta (4 \cos^2 \theta - 3 - 2 \sin \theta) = 0\]
\[\Rightarrow 4 \cos^2 \theta - 3 - 2 \sin \theta = 0\]
\[\Rightarrow 4(1 - \sin^2 \theta) - 3 - 2 \sin \theta = 0\]
\[\Rightarrow 4 \sin^2 \theta + 2 \sin \theta - 1 = 0\]

Now solve this quadratic equation in \( \sin \theta \) to find the exact value of \( \sin \theta \). Compare your answer to the answer obtained in the previous example.

**Exercise.** Find the exact value of \( \cos 36^\circ \).

**Exercise.** Show that for any number \( \theta \),
\[
\cos^3 \theta - \sin^3 \theta = (\cos \theta - \sin \theta)(1 + \cos \theta \sin \theta)
\]
for all numbers \( \theta \).

**Exercise.** Show that for any number \( \theta \),
\[
(\cos \theta + \sin \theta)^2 = 1 + \sin 2\theta
\]
for all numbers \( \theta \).

**Example.** Prove the following identity.
\[
\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}
\]
where \( \theta \) is a real number.

**Answer.** The identity is invalid when \( \theta = (2n + 1)\pi \), for any integer \( n \). Therefore, let us assume otherwise.

**Left Side** = \( \cos \theta \)

\[= \cos(2(\theta/2))\]
\[ = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \]
\[ = \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}} \]
\[ = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}, \]

by dividing both the numerator and the denominator by \( \cos^2 \frac{\theta}{2} \).

**Exercise.** Prove the following identity.

\[ \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \]

where \( \theta \) is a number.
Chapter 12

Applications: Trigonometric Equations

Earlier we obtained general solutions to simple trigonometric equations \( \sin \theta = a \), \( \cos \theta = a \), and \( \tan \theta = a \), where \( a \) is a number and \( -1 \leq a \leq 1 \), in the first two cases. The following are those results.

**General Solution to the Sine Equation Theorem.**

Consider the equation \( \sin x = a \), where \( a \) is a real number.

1. If \( a > 1 \) or \( a < -1 \), then the equation has no solutions.
2. If \( -1 \leq a \leq 1 \), then the general solution of the equation is \( n\pi + (-1)^n \sin^{-1} a \), where \( n \) is any integer.

**General Solution to the Cosine Equation Theorem.**

Consider the equation \( \cos x = a \), where \( a \) is a real number.

1. If \( a > 1 \) or \( a < -1 \), then the equation has no solutions.
2. If \( -1 \leq a \leq 1 \), then the general solution of the equation is \( 2n\pi \pm \cos^{-1} a \), where \( n \) is any integer.
General Solution to the Tangent Equation Theorem.
Consider the equation $\tan x = a$, where $a$ is a real number.
Then the general solution of the equation is $n\pi + \tan^{-1}a$, where $n$ is any integer.

With the help of trigonometric and algebraic identities we can solve many more trigonometric equations. In this section we will look at some of those methods.

**Example.** Solve $\sin^2\theta - 2\sin\theta - 3 = 0$.

**Answer.** Notice that this is a quadratic equation in $\sin\theta$. Assume that this equation is true for some number $\theta$. The left side of this equation can be factored into a product of two linear factors of $\sin\theta$. That is,

$$(\sin\theta - 3)(\sin\theta + 1) = 0.$$ 

By the Zero Product Property Theorem, either $\sin\theta - 3 = 0$ or $\sin\theta + 1 = 0$. As a result, we have two linear equations in $\sin\theta$. Namely, $\sin\theta = 3$ and $\sin\theta = -1$.

By the General Solution to the Sine Equation Theorem, the equation $\sin\theta = 3$ has no solutions. By the same theorem, the general solution to the second equation $\sin\theta = -1$ is:

$$\theta = n\pi + (-1)^n\sin^{-1}(-1), \text{ where } n \text{ is an integer}.$$ 

$$= n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$$

$$= n\pi - (-1)^n\left(\frac{\pi}{2}\right).$$

If $n$ is odd, then the general solution is $\theta = n\pi + \frac{\pi}{2}$ and if $n$ is even, then the general solution is $\theta = n\pi - \frac{\pi}{2}$. In any of these cases, $\sin\theta = -1$. Therefore, by substituting $-1$ for $\sin\theta$ in the given equation you can see that all these solutions check out.

Therefore, the general solution to the given equation is $n\pi - (-1)^n\left(\frac{\pi}{2}\right)$, where $n$ is an integer.

□
Example. Solve $2 \cos^2 \theta - 1 = 0$.

Answer. This is a quadratic equation in $\cos \theta$. By using the difference of squares identity, we can factor the left side of this equation to obtain

$$(\sqrt{2} \cos \theta - 1)(\sqrt{2} \cos \theta + 1) = 0.$$ 

Then by the Zero Product Property Theorem, either $\sqrt{2} \cos \theta - 1 = 0$ or $\sqrt{2} \cos \theta + 1 = 0$. Therefore,

$$\cos \theta = \frac{1}{\sqrt{2}} \text{ or } \cos \theta = -\frac{1}{\sqrt{2}}.$$ 

By the General Theorem to the Cosine Equation Theorem, the general solution to the first equation is:

$$\theta = 2n\pi \pm \cos^{-1}\left(\frac{1}{\sqrt{2}}\right), \text{ where } n \text{ is an integer.}$$

$$= 2n\pi \pm \frac{\pi}{4}.$$ 

For any $n$, $\cos(2n\pi \pm \frac{\pi}{4}) = \cos(\pm \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. By substituting $\frac{1}{\sqrt{2}}$ in the original equation, you can see that the original equation is true for any of these numbers.

By the General Theorem to the Cosine Equation Theorem, the general solution to the second equation is:

$$\theta = 2n\pi \pm \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right), \text{ where } n \text{ is an integer.}$$

$$= 2n\pi \pm \frac{3\pi}{4}.$$ 

For any $n$, $\cos(2n\pi \pm \frac{3\pi}{4}) = \cos(\pm \frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$. By substituting $-\frac{1}{\sqrt{2}}$ in the original equation you can see that the original equation is true for any of these numbers.

Therefore, the general solution to the given equation is:

$$2n\pi \pm \frac{\pi}{4} \text{ or } 2n\pi \pm \frac{3\pi}{4}, \text{ where, } n \text{ is an integer.} \quad \Box$$ 

You can also solve the equation in the previous example if you recognize that the left side of the equation is $\cos 2\theta$. This is because, $\cos 2\theta = 2 \cos^2 \theta - 1$. Let us solve the previous equation now using this identity.
Example. Solve $2 \cos^2 \theta - 1 = 0$.

Answer. Assume the given equation is true for some number $\theta$.

\[
2 \cos^2 \theta - 1 = 0
\]
\[
\implies \cos 2\theta = 0
\]
\[
\implies 2\theta = 2n\pi \pm \cos^{-1} 0, \text{ where } n \text{ is an integer.}
\]
\[
\implies 2\theta = 2n\pi \pm \frac{\pi}{2}
\]
\[
\implies \theta = n\pi \pm \frac{\pi}{4}
\]

If $n$ is odd, then $\cos(n\pi \pm \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$. The given equation is true for all those numbers. If $n$ is even, then $\cos(n\pi \pm \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. Again, the given equation is true for all those numbers.

Therefore, the general solution of the given equation is $n\pi \pm \frac{\pi}{4}$, where $n$ is an integer.

Exercise. Find particular solutions to the equation $2 \cos^2 \theta - 1 = 0$ in the interval $[0, 2\pi]$, first, using the general solution found in the first method, and second, using the general solution found in the second method.

Based on our experience with solving algebraic equations, we know that we sometimes get extraneous solutions. One instance, for example, was when we used the theorem: If $a = b$ then $a^2 = b^2$, where $a$ and $b$ are real numbers. In the following example, we could use this theorem. However, we will have to carefully check the solutions as some of the solutions can be extraneous solutions.

Example. Find the general solution to the equation $\sin \theta - \cos \theta = 1$.

Answer (Method 1). Assume the given equation is true for some number $\theta$.

If we square both sides of the equation, then

\[
\sin \theta - \cos \theta = 1
\]
\[
\implies (\sin \theta - \cos \theta)^2 = 1
\]
\[
\implies \sin^2 \theta - 2 \sin \theta \cos \theta + \cos^2 \theta = 1
\]
\[
\implies 1 - \sin 2\theta = 1
\]
\[
\implies \sin 2\theta = 0
\]

\footnote{See the section on solving radical equations for an example.}
\[ \Rightarrow \quad 2\theta = n\pi + (-1)^n \sin^{-1} 0, \quad \text{where } n \text{ is an integer.} \]

\[ \Rightarrow \quad 2\theta = n\pi \]

\[ \Rightarrow \quad \theta = \frac{n\pi}{2} \]

If \( n = 4k + 1 \), where \( k \) is any integer, then \( \sin \left( \frac{n\pi}{2} \right) = 1 \) and \( \cos \left( \frac{n\pi}{2} \right) = 0 \), and the given equation IS true.

If \( n = 4k - 1 \), where \( k \) is any integer, then \( \sin \left( \frac{n\pi}{2} \right) = -1 \) and \( \cos \left( \frac{n\pi}{2} \right) = 0 \), and the given equation IS NOT true.

If \( n = 4k \), where \( k \) is any integer, then \( \sin \left( \frac{n\pi}{2} \right) = 0 \) and \( \cos \left( \frac{n\pi}{2} \right) = 1 \), and the given equation IS NOT true.

If \( n = 4k + 2 \), where \( k \) is any integer, then \( \sin \left( \frac{n\pi}{2} \right) = 0 \) and \( \cos \left( \frac{n\pi}{2} \right) = -1 \), and the given equation IS true.

Therefore, the general solution of the given equation is \( \frac{n\pi}{2} \), where \( n \) is an integer of the form \( 4k + 1 \) or \( 4k + 2 \), where \( k \) is any integer. That is, all solutions of the given equation can be written as \( 2k\pi + \frac{\pi}{2} \) or \( 2k\pi + \pi \), where \( k \) is any integer. \( \square \)

The struggle to find the general solution of the equation of \( \sin \theta - \cos \theta = 1 \) as displayed above leads us to seek a different method to find the general solution. One alternative method of finding the general solution without “squaring both sides” is given below. The plan is to use one of the super-star identities. We will develop the general method first and then look at the previous example again.

**Example.** Find the general solution to the equation \( a\sin \theta - b\cos \theta = c \), where \( a, b \) are positive real numbers and \( c \) is a real number.

**Answer.** Assume the given equation is true for some number \( \theta \).

Divide both sides of the equation by \( \sqrt{a^2 + b^2} \). Then the given equation becomes:

\[
\frac{a}{\sqrt{a^2 + b^2}} \sin \theta + \frac{b}{\sqrt{a^2 + b^2}} \cos \theta = \frac{c}{\sqrt{a^2 + b^2}}
\]

First, notice that

\[
\left( \frac{a}{\sqrt{a^2 + b^2}} \right)^2 + \left( \frac{b}{\sqrt{a^2 + b^2}} \right)^2 = 1
\]
Therefore, \( P \left( \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right) \) is a point on the unit circle. In addition, since both coordinates of \( P \) are positive, \( P \) is in the first quadrant. Then by the definitions of sine and cosine numbers of an angle, there is an angle \( \alpha \) in the first quadrant so that

\[
\cos \alpha = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin \alpha = \frac{b}{\sqrt{a^2+b^2}}
\]

Therefore, the given equation can be written as:

\[
\sin \theta \cos \alpha - \cos \theta \sin \alpha = \frac{c}{\sqrt{a^2+b^2}}
\]

By using one of the super-star identities, we can write the above equation as:

\[
\sin(\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}}
\]

If \(-1 \leq \frac{c}{\sqrt{a^2+b^2}} \leq 1\), then the above equation has solutions, and the general solution is:

\[
\theta - \alpha = n\pi + (-1)^n \sin^{-1} \left( \frac{c}{\sqrt{a^2+b^2}} \right), \quad \text{where} \quad n \quad \text{is any integer.}
\]

Therefore, the general solution to the given equation is:

\[
\theta = n\pi + (-1)^n \sin^{-1} \left( \frac{c}{\sqrt{a^2+b^2}} \right) + \alpha. \quad \Box
\]

Now we will use the method discovered in the previous example to find the general solution of \( \sin \theta - \cos \theta = 1 \).

**Example.** Find the general solution to the equation \( \sin \theta - \cos \theta = 1 \).

**Answer (Method 2).** Assume the given equation is true for some number \( \theta \).

Divide both sides of the equation by \( \sqrt{1^2 + 1^2} = \sqrt{2} \). Then

\[
\sin \theta - \cos \theta = 1 \implies \frac{1}{\sqrt{2}} \sin \theta - \frac{1}{\sqrt{2}} \cos \theta = \frac{1}{\sqrt{2}} \\
\implies \sin \theta \cos \frac{\pi}{4} - \cos \theta \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \implies \sin \left( \theta - \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} \implies \theta - \frac{\pi}{4} = n\pi + (-1)^n \sin^{-1} \left( \frac{1}{\sqrt{2}} \right), \quad \text{where} \quad n \quad \text{is an integer.}
\[ \theta - \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4} \]
\[ \Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4} \]

If \( n \) is even, that is \( n = 2k \) for some integer \( k \), then \( \theta = 2k\pi + \frac{\pi}{2} \) and if \( n \) is odd, that is, if \( n = 2k + 1 \) for some integer \( k \), then \( \theta = (2k + 1)\pi \). By our previous work, we know that both sets are solutions of the given equation. Therefore, the general solution of the given equation is \( n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4} \), where \( n \) is any integer.

\[ \square \]

**Exercise.** Find the general solution of \( \sin \theta + \cos \theta = \sqrt{2} \), where \( \theta \) is a real number.
Consider a particle moving around a circle of radius $r$. We say the particle is in a *circular motion*. Assume further that this particle moves at a *constant speed*. That is, the average speed of the particle over *any* given time interval is constant. What does that mean precisely? If $t_1$ and $t_2$ are the lengths of two arbitrary time intervals (in time units) and $s_1$ and $s_2$ are the distance travelled by the particle (in distance units) in those time intervals respectively, then $\frac{s_1}{t_1} = \frac{s_2}{t_2}$. Let us call this constant speed $v$ “distance units per time unit”. Then

$$v = \frac{s_1}{t_1} = \frac{s_2}{t_2} = \frac{s}{t},$$

where $s$ is the distance travelled in any given time interval $t$. In the case of circular motion, $s$ is the arc length of a sector of some central angle $\theta$. If we use the radian measure for $\theta$, then by definition,

$$s = r\theta$$

Therefore,

$$v = \frac{r\theta}{t}$$

Since $v$ and $r$ are constants, $\frac{\theta}{t}$ is a constant. We call this the *constant angular speed of the circular motion*. We will use the Greek letter “omega” to represent the constant angular speed.

$$\omega = \frac{\theta}{t}$$
Therefore, for a circular motion, the constant speed and the constant angular speed are related by

\[ v = r \omega \]

If we know either the constant speed or the constant angular speed, then we can calculate the distance travelled and the angle travelled by the particle in a given period of time.

**Example.** Suppose a particle travels along a circle with radius 5 meters. Suppose the constant speed of the particle is 2 meters/second. Find the constant angular speed of the particle. Find how long it takes the particle to go around the circle once.

**Answer.** Since \( v = r \omega \), \( 2 = 5 \omega \). Therefore, the angular speed of the particle is \( \frac{2}{5} \) radians/second.

To go around the circle once, the particle must travel an angle of \( 2\pi \). Suppose it takes \( t_0 \) seconds for the particle to travel the angle of \( 2\pi \). Then

\[
\omega = \frac{\theta}{t} \\
\Rightarrow \frac{2}{5} = \frac{2\pi}{t_0} \\
\Rightarrow t_0 = \frac{\pi}{\frac{2}{5}}
\]

Therefore, it takes \( \frac{\pi}{\frac{2}{5}} \) seconds for the particle to go around the circle once.

Introduce a coordinate system to the given circular motion so that the center of the circle with radius \( r \) is the origin of the coordinate system. By the definition of sine and cosine functions, the coordinates of the position of the moving particle, for a given angle \( \theta \), are \((r \cos \theta, r \sin \theta)\). That is, for a circular motion with constant angular speed of \( \omega \),

\[ x = r \cos \omega t, \text{ and } y = r \sin \omega t \]

Since \( r \) and \( \omega \) are constants, both \( x \) and \( y \) depend only on time \( t \), and therefore, both \( x \) and \( y \) are functions of time. We indicate this by writing \( x \) and \( y \) using the functional notation as follows.

\[ x(t) = r \cos \omega t, \text{ and } y(t) = r \sin \omega t \]

Now suppose a particle is allowed to move only along the \( y \) axis so that its motion is given by

\[ y = r \sin \omega t \]
Then we say the particle is in a *simple harmonic motion*. Then $r$ is the amplitude of the simple harmonic motion and $\frac{2\pi}{\omega}$ is the period of the simple harmonic motion. The quantity $\frac{\omega}{2\pi}$ is called the *frequency* of the simple harmonic motion.\footnote{You will learn more details about simple harmonic motion when you take a course in the branch of mathematics usually known as differential equations.}

Instances of a spring in a simple harmonic motion

**Exercise.** Suppose a spring is in a simple harmonic motion given by $y(t) = 3\sin 2t$. Find the amplitude, the period, and the frequency of this motion.
Chapter 14

An Introduction to Polar Coordinates

Since high school you have been using a coordinate system known as the Cartesian coordinate system or the rectangular coordinate system. This system is designed to identify any point on the plane by two numbers chosen as follows. Draw two number lines perpendicular to each other so that the point of intersection coincides with the point 0 on each line. We identify one of the number lines as the first number line and the other as the second number line. Usually the first number line is called the $x$-axis and is usually shown as a horizontal line. The second number line is called the $y$-axis and is usually shown as a vertical line. Let $P$ be any arbitrary point on the plane. Drop a perpendicular from $P$ to the $x$-axis. Then this perpendicular will land on a number since the $x$-axis is a number line. Let us call this number $x_p$. Drop a perpendicular from $P$ to the $y$-axis. Suppose this perpendicular lands on the number $y_p$. We will identify the point $P$ using the two numbers $x_p$ and $y_p$ in that order. Conventionally this ordered pair of numbers is written as $(x_p, y_p)$. The number $x_p$ is called the $x$-coordinate of $P$ and the number $y_p$ is called the $y$-coordinate of $P$. We usually say the coordinates of $P$ are $(x_p, y_p)$.  

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The following is another theorem that you may have learned in high school.

**Theorem.** Given a point $P$ and a line $\ell$ on a plane, there is exactly one line passing through $P$ perpendicular to $\ell$ on the plane.

As a result of the above theorem, the coordinates of a point in the rectangular coordinate system are *unique*. Clearly, the coordinates of the origin (the point where the two perpendicular number lines intersect) are $(0,0)$.

We have used the rectangular coordinate system to sketch the graph of a function $f$ by identifying the coordinates of all points $(x,f(x))$ and only those points on the plane. For example, the graph of $f(x) = \sin x$ is the collection of all points $(x,\sin x)$ on the plane.

### 14.1 Polar Coordinate System

We will introduce another useful coordinate system known as the *polar coordinate system* now. Once again consider a plane. Pick a point $O$ on this plane. This fixed point $O$ will be called the *pole* from now on. Draw a ray so that the vertex of the ray is $O$. This fixed ray will be called the *polar axis* from now on.

![Polar Axis](image)

Consider any arbitrary point $P$ on the plane other than $O$. Let $\ell$ be the line passing through $O$ and $P$.

---

1. Through two (distinct) points on the plane passes exactly one line.
Let $\theta$ be an angle (in radians) whose initial side is the polar axis, whose vertex is $O$ and whose terminal side lies on $\ell$. As you can imagine, there are infinitely many choices to pick an angle as described above. Let $r$ be the length of the segment $OP$. Clearly, $r$ is a positive number. Let $\theta$ be a non-negative number. (The value of $\theta$ depends on the angle.) We will assign an ordered pair of numbers to $P$ as follows.

1. Suppose $\theta$ is measured counterclockwise and $P$ lies on the terminal side of $\theta$. Then we assign $(+r,+\theta)$ to $P$.

2. Suppose $\theta$ is measured clockwise and $P$ lies on the terminal side of $\theta$. Then we assign $(+r,-\theta)$ to $P$. 
3. Suppose \( \theta \) is measured counterclockwise and \( P \) does not lie on the terminal side of \( \theta \). Then we assign \((-r, +\theta)\) to \( P \).

4. Suppose \( \theta \) is measured clockwise and \( P \) does not lie on the terminal side of \( \theta \). Then we assign \((-r, +\theta)\) to \( P \).

This system described so far leaves the point \( O \) on the plane with no assigned ordered pair of numbers. Now we will assign an ordered pair of numbers to \( O \). Suppose \( P = O \). Then there are infinitely many lines that pass through \( P \) and \( O \). We can pick any one of those lines to assign an ordered pair for \( O \). Clearly, \( r = 0 \) and \( \theta \) is arbitrary. Therefore,
we will assign \((0, \theta)\) for \(O\), where \(\theta\) is any arbitrary real number.

The system described above is known as the Polar Coordinate System of the plane. An ordered pair \((r, \theta)\) for a given point \(P\) is called the polar coordinates of \(P\).

### 14.2 Polar Equations and Graphs

A polar equation is an equation in \(r\) and \(\theta\). The following are a few examples of polar equations.

\[
\begin{align*}
r &= 1 \\
\theta &= \frac{\pi}{4} \\
r &= \theta
\end{align*}
\]

The graph of a polar equation is the collection of all solutions \((r, \theta)\) of the given equation. We can sketch graphs of polar equations in polar coordinates by using few selected solutions and making an educated judgment of the placement of the rest of the solutions on the plane as you probably did when you first looked at graphs of equations in \(x\) and \(y\) in high school.

**Example.** Sketch the graph of \(r = 1\).

Let us try to sketch the graph of the equation \(r = 1\). We notice that there is no \(\theta\) in this equation. That means \(\theta\) is arbitrary. That is, for any \(\theta\), \((1, \theta)\) is a solution of the given equation. Let us select the radian measures of special angles for \(\theta\) and make a table of selected solutions as you may have done in high school when you first leaned how to sketch a graph of an equation.

<table>
<thead>
<tr>
<th>(r)</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{6})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{4})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{3})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>(\pi)</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{3\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{5\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{7\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{9\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{11\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>(2\pi)</td>
</tr>
</tbody>
</table>

We will sketch these points on a plane using the polar coordinate system.
CHAPTER 14. AN INTRODUCTION TO POLAR COORDINATES

Now we can make an educated guess that the graph of the equation \( r = 1 \) is a circle with center \( O \) and radius 1.

**Example.** Sketch the graph of \( \theta = \frac{\pi}{4} \).

We will make a table again. In this case, \( r \) is arbitrary. We will select a few positive integer values, a few negative integer values, and 0 for \( r \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{\pi}{4} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>-2</td>
<td>( \frac{\pi}{4} )</td>
</tr>
<tr>
<td>-3</td>
<td>( \frac{\pi}{4} )</td>
</tr>
</tbody>
</table>
We can see that the graph of the $\theta = \frac{\pi}{4}$ is a line passing through $O$ with slope $= 1$.

**Exercise.** Sketch the graph of $r = \theta$.

**Example.** Sketch the graph of $r = \cos \theta$.

Once again we will use the special angles to create a table of points and then sketch those points to predict the graph of $r = \cos \theta$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\pi}{6}$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{\pi}{3}$</td>
</tr>
<tr>
<td>0</td>
<td>$\frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{1}{2}$</td>
<td>$\frac{2\pi}{3}$</td>
</tr>
<tr>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$\frac{3\pi}{4}$</td>
</tr>
<tr>
<td>$-\frac{\sqrt{3}}{2}$</td>
<td>$\frac{5\pi}{6}$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>
It looks like the graph of \( r = \cos \theta \) is a circle with radius \( \frac{1}{2} \) and the center at a distance of \( \frac{1}{2} \) from \( O \) on the polar axis.

### 14.3 Finding Relationships between Polar Coordinates and Rectangular Coordinates

Let us introduce the rectangular coordinate system on the same plane with the polar coordinate system as follows. We will select the origin of the rectangular coordinate system as the pole \( O \), we will select the \( x \)-axis so that the polar axis is on the \( x \)-axis, and the \( y \)-axis passes through \( O \).

Let \( P \) be an arbitrary point other than \( O \) on the plane. Suppose the rectangular coordinates of \( P \) are \((x, y)\) and a polar coordinate of \( P \) are \((r, \theta)\). Then by the definitions of
14.3. RELATIONSHIPS BETWEEN POLAR AND RECTANGULAR COORDINATES

\[ \cos \theta \text{ and } \sin \theta, \cos \theta = \frac{x}{r} \text{ and } \sin \theta = \frac{y}{r}. \] Therefore, \( x = r \cos \theta \) and \( y = r \sin \theta \). By the distance formula, \( |OP| = \sqrt{x^2 + y^2} \). Since \( \frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} \) and since \( r \neq 0 \), we get, \( \tan \theta = \frac{y}{x} \).

These are the relationships that we sought.

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
r &= \pm \sqrt{x^2 + y^2} \\
\tan \theta &= \frac{y}{x}
\end{align*}
\]

Clearly, the rectangular coordinates of \( O \) are \((0, 0)\) and \( r = 0 \), and \( \theta \) is arbitrary at \((0, 0)\).

**Example.** Convert the equation \( r = 1 \) to rectangular coordinates.

**Answer.** In this equation, \( \theta \) is arbitrary. Since \( r = \pm \sqrt{x^2 + y^2} \), we can write \( r = 1 \) as \( \pm \sqrt{x^2 + y^2} = 1 \). This is the same as \( x^2 + y^2 = 1 \).

Earlier we suspected the graph of \( r = 1 \) to be the circle with center \( O \) and radius 1. Since the graph of \( x^2 + y^2 = 1 \) is the unit circle, this confirms that our guess was correct.

**Example.** Convert the equation \( \theta = \frac{\pi}{4} \) to rectangular coordinates.

**Answer.** In this equation \( r \) is arbitrary. Since \( \tan \frac{\pi}{4} = 1 \), we can write \( \theta = \frac{\pi}{4} \) as \( \frac{y}{x} = 1 \). If \( x \neq 0 \), then \( \frac{y}{x} = 1 \) is the same as \( y = x \). However, \( O \) is on the graph of \( r = \tan \frac{\pi}{4} \) since \((0, \frac{\pi}{4})\) are polar coordinates of \( O \). Therefore by including the solution \((0,0)\) in the existing set of solutions, we get \( y = x \) with no restrictions. Therefore, \( \theta = \frac{\pi}{4} \) in rectangular coordinates is \( y = x \).
Earlier we suspected that the graph of $\theta = \frac{\pi}{4}$ to be the line passing though $O$ with slope 1. Now we know that our guess was correct.

**Example.** Convert the equation $r = \cos \theta$ to rectangular coordinates.

**Answer.** If we multiply both sides of the equation by $r$, then we get $r^2 = r \cos \theta$. By using conversion formulas, this equation becomes $x^2 + y^2 = x$. \(\square\)

Earlier we suspected that the graph of $r = \cos \theta$ to be the circle with the center $\frac{1}{2}$ from $O$ on the polar axis and radius $\frac{1}{2}$. This suspicion should lead us to expect this graph to be the same as the equation of a circle with center $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ in rectangular coordinates. Is this prediction correct?

\[
x^2 + y^2 = x \\
\implies x^2 - x + y^2 = 0 \\
\implies x^2 - x + \left(\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2 \\
\implies \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2
\]

This is indeed the equation of the circle with center $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. (We used the completing the square method in the above calculation.)

**Exercise.** Convert the equation $r = \cos \theta$ to rectangular coordinates.

### 14.4 Selecting a Unique Pair of Polar Coordinates for a Point

In contrast to the rectangular coordinates of a point, there are infinitely many polar coordinates for a given point. It would be nice if we could select one pair of polar coordinates as the standard polar coordinates of a point. By introducing restrictions to $r$ and $\theta$, we can obtain a unique pair of polar coordinates for a point. First we introduce the restriction $r \geq 0$. That is, we will always select the angle so that $P$ lies on the terminal side of $\theta$. In other words, we eliminate possibilities (3) and (4) in the description of the polar coordinate system. (See the beginning of the section titled “The Polar Coordinate System”.)
Next we introduce the restriction $\theta \geq 0$. That is we will measure angles only in the counterclockwise direction. This eliminates possibility (2). The two restrictions introduced thus far leave only the first option to select polar coordinates for a point. However, there are still infinitely many ways to select polar coordinates for a given point $P$. The following are two such possibilities.

To eliminate all the infinitely many possible polar coordinates of $P$ except one, we will add another restriction: $\theta < 2\pi$. With the following restrictions, any given point $P$ will have a unique ordered pair of polar coordinates.

$$r \geq 0 \text{ and } 0 \leq \theta < 2\pi.$$ 

With the above restrictions in place, we can state the relationships between rectangular coordinates and polar coordinates as follows.

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  r &= \sqrt{x^2 + y^2} \\
  \tan \theta &= \frac{y}{x}
\end{align*}
\]

Under these new restrictions, however, the graph of $\theta = \frac{\pi}{4}$ is no longer a line passing through $O$ with slope 1.

**Exercise.** What is the graph of $\theta = \frac{\pi}{4}$ under the restrictions $r \geq 0$ and $0 \leq \theta < 2\pi$?
Chapter 15

An Introduction to Complex Plane

Consider a number line. Any point on this line is a real number. We will call this number line “the real number line” or the “real axis”. Now imagine “rotating a number (a point on the real axis) by \( \frac{\pi}{2} \) counterclockwise about 0”. We will indicate this rotation by \( i \). Then the notation \( 2i \) means “2 is rotated by \( \frac{\pi}{2} \) counterclockwise about 0”. Suppose we rotate all real numbers this way. Then we will get a line perpendicular to the real axis at 0. We will call this new axis the imaginary axis. Each point \( a \) on the real axis corresponds to the point \( ai \) on the imaginary axis. In that sense the imaginary axis is a “number line” with unit \( i \). We will call these numbers imaginary numbers and we will call \( i \) the imaginary unit. The real axis and the imaginary axis together define a plane.\footnote{Recall that through any two distinct lines passes a unique plane.} We will call this plane the complex plane. The point on the complex plane where the real axis and the imaginary axis meet is called the origin.
We define imaginary number addition as follows.

\[ ai + bi = (a + b)i \]

That is, \( ai + bi \) is the concatenation of imaginary segments \( ai \) and \( bi \) of lengths \( a \) and \( b \) respectively.\(^2\)

We define imaginary number multiplication as follows.

\[ (ai)(bi) = (ab)ii \]

Notice that \( ii \) is “rotate \( i \) about the origin by \( \frac{\pi}{2} \) counterclockwise.” That is, \( ii = -1 \). We denote this as \( i^2 = -1 \). Also \( i^2i = (-1)i \) is \(-i\), and we denote this as \( i^3 = -i \). Continuing in this fashion, \( i^4 = 1 \), \( i^5 = i \), and so on.

Multiplication of a real number \( a \) and \( i \) is defined as \( ai \). That is \( a \cdot i = ai \). Therefore, \( i = 1 \cdot i \).

\(^2\)This is analogous to the real number addition of \( a + b \) as the concatenation of segments of lengths \( a \) and \( b \) respectively.
Now we define the addition of a real number and an imaginary number $z = a + bi$ as follows. Go to the real number $a$. That is, go to that point on the real axis. Then from there rotate the real number $b$ by $\frac{\pi}{2}$ counterclockwise about the point $a$. Then the resulting point on the complex plane is called the *complex number* $a + bi$. In this fashion we can identify each point on the complex plane by a complex number. Notice that both $a$ and $b$ are real numbers. The real number $a$ is called the *real part* of $z$ and the real number $b$ is called the *imaginary part* of $z$.

![Diagram of complex plane with point a and vector bi leading to z = a + bi](image)

The real numbers are the complex numbers of the form $a + 0i$ and the imaginary numbers are the complex numbers of the form $0 + bi$. By this definition of a complex number $i = 0 + i$, $1 = 1 + 0i$ and $0 = 0 + 0i$.

We say two complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ are *equal* if they are the *same point* on the complex plane. That is, $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$. If $z = a + bi$ is a complex number, then $z = 0$ if and only if both $a = 0$ and $b = 0$.

Since the collection of the real numbers is a subset of the complex numbers, we want the complex numbers to have the same nice properties the real numbers have; namely, associative properties, commutative properties (collecting like terms) and the distributive property.

The *addition* of two complex numbers $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$ denoted by $z_1 + z_2$ is

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i.$$
The multiplication of the two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ denoted by $z_1 z_2$ is

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i.$$  

You should be able to verify this using the distributive property, collecting like terms and using $i^2 = -1$.

The subtraction of $z_2$ from $z_1$ is denoted by $z_1 - z_2$ and it is

$$z_1 - z_2 = (a_1 - a_2) - (b_1 - b_2)i.$$  

Suppose $z = a + bi \neq 0$. We will define $\frac{1}{z}$ as the complex number so that $z \cdot \frac{1}{z} = 1$. You can see that $(a + bi)(a - bi) = a^2 + b^2$. The number $a - bi$ denoted by $\overline{z}$ is called the complex conjugate of $z$. Then $z\overline{z}$ is the real number $a^2 + b^2$. Since $z \neq 0$, $a^2 + b^2 \neq 0$. Therefore, we know that $\frac{a^2 + b^2}{a^2 + b^2} = 1$. If we denote $z\overline{z}$ as $|z|^2$, then

$$\frac{1}{z} = \frac{a}{|z|^2} - \frac{b}{|z|^2}i.$$  

We will denote $\frac{a}{|z|^2} = \frac{b}{|z|^2}i$ as $\overline{z}$. Then

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$  

Now we can define the division of complex numbers. Suppose $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ are two complex numbers so that $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{|z_2|^2}.$$  

In other words,

$$\frac{z_1}{z_2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) - \left(\frac{a_1 b_2 - b_1 a_2}{a_2^2 + b_2^2}\right)i.$$  

Exercise. Verify the above assertion.

15.1 Polar Form of a Complex Number

Let us introduce a polar coordinate system in the complex plane by choosing the point $0 + 0i$ as the pole and the positive real axis as the polar axis. Consider an arbitrary
complex number \( z = a + bi \). Let \((r, \theta)\) be the polar coordinates of \( z \) as seen as a point with rectangular coordinates \((a, b)\). We will introduce the standard restrictions on \( r \) and \( \theta \), that is \( r \geq 0 \) and \( 0 \leq \theta < 2\pi \), to make the polar coordinates of \( z \) unique. Then

\[
    r = \sqrt{a^2 + b^2}, \quad a = r \cos \theta, \quad b = r \sin \theta.
\]

Therefore, \( z \) can be written as

\[
    z = r (\cos \theta + i \sin \theta), \quad \text{where, } r \geq 0 \text{ and } 0 \leq \theta < 2\pi.
\]

This is known as the standard polar form of a complex number. The number \( r \) is called the modulus of the complex number and the number \( \theta \) is called the argument of the complex number. Notice that modulus of \( z \) is \(|z|\). We abbreviate the modulus of \( z \) as “mod \( z \)” and the argument of \( z \) as “arg \( z \)”.

**Example.** Write \( z = 2 - 2i \) in standard polar form.

**Answer.** \( r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2} \). Therefore, \( 2\sqrt{2} \cos \theta = 2 \), and \( 2\sqrt{2} \sin \theta = -2 \). Since \( \sin \theta < 0 \) and \( \cos \theta > 0 \), the angle \( \theta \) is in the fourth quadrant. \( 2\sqrt{2} \cos \theta = 2 \) implies that \( \cos \theta = \frac{1}{\sqrt{2}} \) and \( \sin \theta = -\frac{1}{\sqrt{2}} \). Therefore, \( \theta = \frac{7\pi}{4} \), and the polar form of the given complex number is

\[
    z = 2\sqrt{2} \left( \cos \left( \frac{7\pi}{4} \right) + i \sin \left( \frac{7\pi}{4} \right) \right). \quad \square
\]

### 15.2 Multiplication of complex numbers in Polar Form

Let \( z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \) be two complex numbers in standard polar form. Then

\[
    z_1 z_2 = (r_1 (\cos \theta_1 + i \sin \theta_1))(r_2 (\cos \theta_2 + i \sin \theta_2))
    = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)
    = r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)
    = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).
\]

Therefore, the modulus of \( z_1 z_2 \) is the product of the moduli and the argument of \( z_1 z_2 \) is the sum of the arguments. We summarize our findings in the following theorem.
CHAPTER 15. AN INTRODUCTION TO COMPLEX PLANE

Complex Number Multiplication Theorem. If \( z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \) are two complex numbers, then \( z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \).

Example. Let \( z_1 = (\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}) \) and \( z_2 = (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) \) be two complex numbers. Find \( z_1 z_2 \) and write it in standard polar form.

Answer. In this problem the mod \( z_1 \) is 1 and mod \( z_2 \) is also 1. Therefore, the product of the modules is 1. The arg \( z_1 \) is \( \frac{11\pi}{6} \) and arg \( z_1 \) is \( \frac{3\pi}{4} \). Therefore the sum of the arguments is \( \frac{11\pi}{6} + \frac{3\pi}{4} \). That is, the sum of the arguments is \( \frac{31\pi}{12} \). However, \( \frac{31\pi}{12} > 2\pi \). So we choose the angle in \( 0 \leq \theta < 2\pi \) which has the same terminal side of \( \frac{31\pi}{12} \). That is, \( \theta = \frac{7\pi}{12} \). Therefore, the standard polar form of \( z_1 z_2 \) is \( \cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12} \).

15.3 Division of complex numbers in Polar Form

Let \( z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \) be two complex numbers in standard polar form. Then

\[
\frac{z_1}{z_2} = \frac{z_1 z_2}{|z_2|^2} = \frac{(r_1 (\cos \theta_1 + i \sin \theta_1)(r_2 (\cos \theta_2 - i \sin \theta_2)))}{r_2^2 (\cos^2 \theta_2 + \sin^2 \theta_2)} = \frac{r_1}{r_2} (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2) = \frac{r_1}{r_2} (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2) = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))
\]

Therefore, the modulus of \( \frac{z_1}{z_2} \) is the quotient of the moduli and the argument of \( \frac{z_1}{z_2} \) is the difference of the arguments. We summarize our findings in the following theorem.

Complex Number Division Theorem. If \( z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \) are two complex numbers, then \( \frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)) \).
Example. Let \( z_1 = (\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) \) and \( z_2 = (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) \) be two complex numbers. Find \( z_1z_2 \) and write it in the standard polar form.

Answer. In this problem the mod \( z_1 \) is 1 and mod \( z_2 \) is also 1. Therefore, the quotient of the modules is 1. The arg \( z_1 \) is \( \frac{\pi}{6} \) and arg \( z_2 \) is \( \frac{3\pi}{4} \). Therefore the difference of the arguments is \( \frac{\pi}{6} - \frac{3\pi}{4} \). That is, the difference of the arguments is \( -\frac{7\pi}{12} \). However, \( -\frac{7\pi}{12} > 2\pi \). So we choose the angle in \( 0 \leq \theta < 2\pi \) which has the same terminal side of \( -\frac{7\pi}{12} \). That is, \( \theta = \frac{17\pi}{12} \). Therefore, the standard polar form of \( \frac{z_1}{z_2} \) is \( \cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \).
Left side of (2) = \((\cos \theta + i \sin \theta)^{k+1}\)
= \((\cos \theta + i \sin \theta)^k(\cos \theta + i \sin \theta)\)
= \((\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta)\), by (1).
= \(\cos(k\theta + \theta) + i \sin(k\theta + \theta)\),
by the Complex Number Multiplication Theorem.
= \(\cos(k + 1)\theta + i \sin(k + 1)\theta\)
= Right side of (2).

Therefore, by mathematical induction, the given statement is true for any positive integer \(n\).

**Case 2:** Suppose \(n = 0\).

We have to show that \((\cos \theta + i \sin \theta)^0 = \cos 0\theta + i \sin 0\theta\).
The left side is 1, by definition, and the right side is \(1 + 0i = 1\). Therefore, the given statement is true for \(n = 0\).

**Case 3:** Suppose \(n < 0\).

Let \(n = -m\). Then \(m\) is a positive integer.

\[
(\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} = \frac{1}{(\cos \theta + i \sin \theta)^m}
= \frac{1}{\cos m\theta + i \sin m\theta}, \text{ by Case 1 of this theorem.}
= \frac{\cos m\theta - i \sin m\theta}{(\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}
= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}
= \frac{\cos m\theta - i \sin m\theta}{\cos m\theta - i \sin m\theta}
= 1
= \cos(-m\theta) + i \sin(-m\theta)
= \cos(-m\theta) + i \sin(-m)\theta
= \cos n\theta + i \sin n\theta
\]

**Example.** Let \(z_1 = 2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})\). Find \((z_1)^{10}\) and write it in the standard polar form.
15.5. DISTINCT COMPLEX $n$TH ROOTS OF A COMPLEX NUMBER IN POLAR FORM

Answer.

$$z_1^{10} = \left[ 2 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \right]^{10}$$

$$= 2^{10} \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)^{10}$$

$$= 2^{10} \left( \cos \frac{50\pi}{6} + i \sin \frac{50\pi}{6} \right), \text{ by De Moivre’s Theorem.}$$

$$= 2^{10} \left( \cos \left( 8\pi + \frac{2\pi}{6} \right) + i \sin \left( 8\pi + \frac{2\pi}{6} \right) \right)$$

$$= 2^{10} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right). \Box$$

15.5 Distinct Complex $n$th roots of a Complex Number in Polar Form

In the eighth grade you may have learned the definition of the $n$th root of a positive real number: if $a$ is a positive real number, then the real number $b$ is called the $n$th root of $a$ if $b^n = a$, where $n$ is a positive integer. We will define the $n$th root of a complex number the same way. If $z$ is a complex number then we say the complex number $w$ is a complex $n$th root of $z$ if $w^n = z$, where $n$ is a positive integer.

Let $z = r(\cos \theta + i \sin \theta)$ be a complex number so that $0 \leq \theta < 2\pi$. Then clearly, $w = r^{1/n}(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n})$ is a complex $n$th root of $z$. (You can verify this by using De Moivre’s Theorem and the definition of the complex $n$th root.)

For $k \geq 0, 2k\pi + \theta < 2(k+1)\pi$, since $\theta < 2\pi$. We also know that $r(\cos(2k\pi + \theta) + i \sin(2k\pi + \theta)) = r(\cos \theta + i \sin \theta) = z$. Therefore, for $k \geq 0$, all $w_k = r^{1/n}(\cos(\frac{2k\pi+\theta}{n}) + i \sin(\frac{2k\pi+\theta}{n}))$ are complex $n$th roots of $z$, where $n > 2$. If $\frac{2k\pi+\theta}{n} < 2\pi$ for some values of $k$, then those $w_k$’s are distinct complex numbers in standard polar form. Therefore, it is possible for $z$ to have many distinct complex $n$th roots for $n > 2$, provided $\frac{2k\pi+\theta}{n} < 2\pi$.

$$\frac{2k\pi+\theta}{n} < 2\pi \implies 2k\pi + \theta < 2n\pi$$

$$\implies 2k\pi < 2n\pi - \theta$$

$$\implies k < n - \frac{\theta}{2\pi}$$

$$\implies k \leq n - 1 \text{ since } k \geq 0 \text{ and } n > 0 \text{ are integers and } \theta < 2\pi.$$
Therefore, \( w_k = r^{1/n}(\cos(\frac{2k\pi+\theta}{n}) + i \sin(\frac{2k\pi+\theta}{n})) \) are distinct complex \( n \)th roots of \( z \) for all \( 0 \leq k \leq n-1 \). We will summarize our findings in the following theorem.

**Complex \( n \)th Roots Theorem.** Let \( z = r(\cos \theta + i \sin \theta) \) be a complex number in standard polar form. Then for a positive integer \( n > 0 \), the distinct complex \( n \)th roots of \( z \) are \( w_k = r^{1/n}(\cos(\frac{2k\pi+\theta}{n}) + i \sin(\frac{2k\pi+\theta}{n})) \), for all integers \( k \), \( 0 \leq k \leq n-1 \).

**Example.** Find all distinct complex 3rd roots of 1.

**Answer.** We can write 1 in standard polar form as follows.

\[
1 = 1 + 0i = \cos 0 + i \sin 0
\]

Now we can find the distinct complex 3rd roots of \( z = \cos 0 + i \sin 0 \) by using the Complex \( n \)th Roots Theorem. The modulus of \( z \) is 1 and \( 1^{1/3} = 1 \). Since \( n = 3 \), \( 0 \leq k \leq 2 \). Then,

\[
w_0 = 1(\cos \frac{0 + 0}{3} + i \sin \frac{0 + 0}{3}) = \cos 0 + i \sin 0 = 1
\]

\[
w_1 = 1 \left( \cos \frac{2\pi + 0}{3} + i \sin \frac{2\pi + 0}{3} \right) = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}
\]

\[
w_2 = 1 \left( \cos \frac{4\pi + 0}{3} + i \sin \frac{4\pi + 0}{3} \right) = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}
\]

Therefore, the distinct complex 3rd roots of 1 are \( 1 \), \( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \), and \( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \). As an exercise, show that \( (\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})^3 = 1 \) and \( (\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3})^3 = 1 \).

A complex equation is an equation containing complex numbers.

**Example.** Find the distinct solutions of the complex equation \( z^4 - 2 = 0 \).

**Answer.** Assume the given equation is true for some complex number \( z \). Then

\[
z^4 = 2
\]

\[
= 2(1 + 0i)
\]

\[
= 2(\cos 0 + i \sin 0)
\]

Now we can find the distinct complex 4th roots of \( z_1 = 2(\cos 0 + i \sin 0) \) by using the Complex \( n \)th Roots Theorem. The modulus of \( z_1 \) is 2, the modulus of any 4th root is \( 2^{1/4} \).

Since \( n = 4 \), \( 0 \leq k \leq 3 \). Then,

\[
w_0 = 2^{1/4}(\cos \frac{0 + 0}{4} + i \sin \frac{0 + 0}{4}) = 2^{1/4}(\cos 0 + i \sin 0) = 2^{1/4}
\]
15.5 DISTINCT COMPLEX nTH ROOTS OF A COMPLEX NUMBER IN POLAR FORM

\[ w_1 = 2^{1/4} \left( \cos \frac{2\pi + 0}{4} + i \sin \frac{2\pi + 0}{4} \right) = 2^{1/4} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \]

\[ w_2 = 2^{1/4} \left( \cos \frac{4\pi + 0}{4} + i \sin \frac{4\pi + 0}{4} \right) = 2^{1/4} \left( \cos \pi + i \sin \pi \right) \]

\[ w_3 = 2^{1/4} \left( \cos \frac{6\pi + 0}{4} + i \sin \frac{6\pi + 0}{4} \right) = 2^{1/4} \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \]

Therefore, the distinct solutions of the given equation are \( 2^{1/4}, 2^{1/4} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right), 2^{1/4} \left( \cos \pi + i \sin \pi \right), \) and \( 2^{1/4} \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right). \) You can use De Moivre’s Theorem to verify that these are actually solutions of the given equation.

Example. Find the distinct solutions of the complex equation \( z^3 - i = 0. \)

Answer. Assume the given equation is true for some complex number \( z. \) Then

\[ z^3 = i \]

\[ = 0 + 1i \]

\[ = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \]

Now we can find the distinct complex 3th roots of \( z = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \) by using the Complex nth Roots Theorem. Then,

\[ w_0 = \left( \cos \left( \frac{0 + \frac{\pi}{2}}{4} \right) + i \sin \left( \frac{0 + \frac{\pi}{2}}{4} \right) \right) = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \]

\[ w_1 = \left( \cos \left( \frac{2\pi + \frac{\pi}{2}}{4} \right) + i \sin \left( \frac{2\pi + \frac{\pi}{2}}{4} \right) \right) = \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8} \]

\[ w_2 = \left( \cos \left( \frac{4\pi + \frac{\pi}{2}}{4} \right) + i \sin \left( \frac{4\pi + \frac{\pi}{2}}{4} \right) \right) = \cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8} \]

Therefore, the distinct solutions of the given equation are \( \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}, \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8}, \) and \( \cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8}. \)

Exercise. Solve the equation \( z^3 = 2 - 2i. \)
Chapter 16

An Introduction to Vectors

A certain physical quantity with two properties, namely, a magnitude and a direction, is called a vector. Examples of such physical quantities are the constant velocity (of a particle moving along a straight line), the constant acceleration (of a particle moving along a straight line), and the weight (of an object on the surface of the Earth). A certain physical quantity with only one property is called a scalar. Examples of scalars are the mass of a particle and the length of a segment.

Geometrically, we will use an arrow with a fixed length to represent a vector.

The point with the arrowhead is called the terminal point of the vector and the point at the other end of the arrow is called the initial point of the vector. If we name the initial point $A$ and the terminal point $B$, then we use the notation $\overrightarrow{AB}$ to represent the vector. We will also use small English letters with an over-line to name vectors. For example, we may use $\mathbf{u}$ or $\mathbf{v}$ to name vectors. (As you may know, we always use small English
CHAPTER 16. AN INTRODUCTION TO VECTORS

letters for numbers.) The direction of a vector \( \overrightarrow{AB} \) is defined by a pair of points: the initial point and the terminal point of \( \overrightarrow{AB} \). If you can imagine going from the initial point \( A \) of a vector \( \overrightarrow{AB} \) to the terminal point \( B \) of \( \overrightarrow{AB} \) along the line segment \( AB \), then you are traveling in the direction of the vector \( \overrightarrow{AB} \). We define the magnitude (or norm) of a vector \( \overrightarrow{AB} \) as the length of the segment \( AB \). We denote the norm of a vector \( \overrightarrow{u} \) by the notation \( \| \overrightarrow{u} \| \).

We define a zero vector denoted by \( \overrightarrow{0} \) as the vector with norm 0. That is, for a zero vector, the initial point and the terminal point are the same. In other words, \( \overrightarrow{AA} \) is a zero vector. We let the zero vector to have any direction it would like to have. In other words, the direction of \( \overrightarrow{0} \) is arbitrary.

Let \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) be two vectors. We say \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) have the same norm if \( \| \overrightarrow{AB} \| = \| \overrightarrow{CD} \| \).

Let \( \overrightarrow{AB} \) be the ray with initial point \( A \) that contains the vector \( \overrightarrow{AB} \).

We say \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) have the same direction if one of the following conditions holds.

1. Suppose \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) lie on the same line. If either \( \overrightarrow{AB} \) contains \( \overrightarrow{CD} \) completely or \( \overrightarrow{CD} \) contains \( \overrightarrow{AB} \) completely, then we say \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) have the same direction.

2. Suppose \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) do not lie on the same line. If \( ABDC \) is a trapezoid with \( AB \parallel CD \) then we say \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) have the same direction.

We say \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) have the opposite direction if one of the following conditions holds.

1. Suppose \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) lie on the same line. If neither \( \overrightarrow{AB} \) contains \( \overrightarrow{CD} \) completely nor \( \overrightarrow{CD} \) contains \( \overrightarrow{AB} \) completely, then we say \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) have the opposite direction.
2. Suppose $\overrightarrow{AB}$ and $\overrightarrow{CD}$ do not lie on the same line. If $ABCD$ is a trapezoid with $AB \parallel DC$ then we say $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the opposite direction.

We say $\overrightarrow{u} = \overrightarrow{v}$ if $\overrightarrow{u}$ and $\overrightarrow{v}$ have the *same* norm and the *same* direction. The “two” vectors in the following figure are the *same*. In other words, we can *move* a vector as long as we do not change any of the two properties of the vector, namely, the norm and the direction.

### 16.1 Vector Addition

Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be vectors. We define $\overrightarrow{u} + \overrightarrow{v}$ as follows. Move the vector $\overrightarrow{v}$ so that its initial point coincides with the terminal point of $\overrightarrow{u}$. The vector with the initial point the same as the initial point of $\overrightarrow{u}$ and the terminal point the same as the terminal point of $\overrightarrow{v}$ is called $\overrightarrow{u} + \overrightarrow{v}$. 
Question. We know that the real numbers satisfy the commutative property of addition. That is, if \(a\) and \(b\) are real numbers, then \(a + b = b + a\). Is vector addition commutative? That is, if \(\mathbf{u}\) and \(\mathbf{v}\) are vectors, then is \(\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}\)?

Answer. If \(\mathbf{u}\) or \(\mathbf{v}\) is \(\mathbf{0}\), then you can easily show that \(\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}\), by using the definition of the vector addition. This is left as an exercise.

Assume \(\mathbf{u} \neq \mathbf{0}\) and \(\mathbf{v} \neq \mathbf{0}\). We can find \(\mathbf{u} + \mathbf{v}\) and \(\mathbf{v} + \mathbf{u}\) according to the definition. Let the initial point of \(\mathbf{u} + \mathbf{v}\) be \(A\), the terminal point of \(\mathbf{u} + \mathbf{v}\) be \(B\), the initial point of \(\mathbf{v} + \mathbf{u}\) be \(C\), and the terminal of \(\mathbf{v} + \mathbf{u}\) be \(D\). We want to show that \(\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}\). That is, we have to show that \(\mathbf{u} + \mathbf{v}\) and \(\mathbf{v} + \mathbf{u}\) have the same norm and the same direction.

You may have learned the following theorem in high school.

The Parallelogram Theorem. A quadrilateral is a parallelogram if and only if a pair of opposite sides are parallel and have the same length.
ABDC is a parallelogram because \( AC = BD \) and \( AC \parallel BD \) by the above theorem. Then by the same theorem, \( AB = CD \) and \( AB \parallel CD \). Therefore, \( \overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u} \).

### 16.2 Scalar Multiplication of a Vector

Suppose \( \overrightarrow{u} \) is a vector and \( k \) is a scalar.

1. If \( \overrightarrow{u} \neq 0 \) and \( k > 0 \), then \( k\overrightarrow{u} \) is the vector with norm \( k\|\overrightarrow{u}\| \) and the direction same as \( \overrightarrow{u} \).
2. If \( \overrightarrow{u} \neq 0 \) and \( k < 0 \), then \( k\overrightarrow{u} \) is the vector with norm \( |k|\|\overrightarrow{u}\| \) and the direction opposite to \( \overrightarrow{u} \).
3. If either \( \overrightarrow{u} = 0 \) or \( k = 0 \), then \( k\overrightarrow{u} = 0 \).

\[
\begin{align*}
\overrightarrow{u} & \quad \frac{1}{2}\overrightarrow{u} \\
\overrightarrow{u} & \quad -\frac{1}{2}\overrightarrow{u}
\end{align*}
\]

**Definition.**

Let \( \overrightarrow{u} \) and \( \overrightarrow{v} \) be two vectors.

1. \(-\overrightarrow{u} = (-1)\overrightarrow{u}\)
2. \(\overrightarrow{u} - \overrightarrow{v} = \overrightarrow{u} + (-1)\overrightarrow{v}\)

A vector \( \overrightarrow{u} \) is called a unit vector if \( \|\overrightarrow{u}\| = 1 \).

**Theorem.** If \( \overrightarrow{u} \) is a non-zero vector given by \( \overrightarrow{u} = \langle u_1, u_2 \rangle \), then \( \frac{1}{\|\overrightarrow{u}\|}\overrightarrow{u} \) is a unit vector.

**Proof.** Since \( \|\overrightarrow{u}\| \neq 0 \), by the definition of the norm of a vector, \( \|\overrightarrow{u}\| > 0 \). Since \( \|\overrightarrow{u}\| \) is a positive scalar, \( \frac{1}{\|\overrightarrow{u}\|} \) is a positive scalar. The norm of the vector \( \frac{1}{\|\overrightarrow{u}\|}\overrightarrow{u} \) is equal to \( \frac{1}{\|\overrightarrow{u}\|}\|\overrightarrow{u}\| \),
by the definition of scalar multiplication. Therefore, the norm of the vector \( \frac{1}{\|\overline{u}\|}\overline{u} \) is 1, (by the cancellation law of the real numbers).

**Exercise.** If \( \overline{u} \) is a non-zero vector, then show that

\[
\|\pi\| \left( \frac{1}{\|\pi\|} \overline{u} \right) = \overline{u}.
\]

With the above exercise done, we are in a position to give meaning to the second property of a non-zero vector \( \overline{u} \), namely the “direction”. Since a vector \( \overline{u} \) has only two properties and \( \|\pi\| \) is the norm, then \( \frac{1}{\|\pi\|} \overline{u} \) must represent the “direction” of the vector.

### 16.3 Algebraic Representation of a Vector

Let \( \overline{u} \) be a vector. Introduce a rectangular coordinate system to the plane. Let \( O \) be the origin of this coordinate system. Move \( \overline{u} \) so that the initial point of \( \overline{u} \) is \( O \). Let \( P(u_1, u_2) \) be the coordinates of the terminal point of \( \overline{u} \). Since the coordinates of \( P \) are unique, we can use the coordinates of \( P \) to give an algebraic representation of \( \overline{u} \).

\[
\overline{u} = \langle u_1, u_2 \rangle
\]

We want to use this new notation for vectors to distinguish the vector \( \overline{u} \) from the point \( P(u_1, u_2) \). The new notation is known as “bra-ket” notation or “angle-rangle” notation.

\[\text{Now we can give an algebraic representation to the norm of a vector. Let } \overline{u} = \langle u_1, u_2 \rangle \text{ be a given non-zero vector. Then by the distance formula,}
\]

\[
\|u\| = \sqrt{u_1^2 + u_2^2}.
\]

\[1\text{Paul Dirac introduced the bra-ket notation in the 1920’s while discovering the principles of quantum physics and Donald Knuth introduced langle-rangle — which stands for left angle and right angle — when he created the \( \text{LaTeX} \) Typesetting System in the 1980’s.}\]
Recall that the unit vector \( \frac{1}{\|u\|} \overline{u} \) gives the direction of a non-zero vector \( \overline{u} \). Therefore,
\[
\frac{1}{\|u\|} \overline{u} = \frac{1}{\sqrt{u_1^2 + u_2^2}} \langle u_1, u_2 \rangle.
\]

We define the unit vector \( \overrightarrow{i} = \langle 1, 0 \rangle \) to be the direction of any vector whose initial point is \( O \) and whose terminal point is on the positive \( x \)-axis.

\[ y \]
\[ \overrightarrow{i} \]
\[ O \]
\[ 1 \]
\[ x \]

**Exercise.** Let \( \overline{u} = \langle 3, 0 \rangle \). Show that \( \overline{u} = 3\overrightarrow{i} \).

\[ y \]
\[ \overline{u} \]
\[ O \]
\[ 3 \]
\[ x \]

**Exercise.** Let \( \overline{v} = \langle -3, 0 \rangle \). Show that \( \overline{v} = -3\overrightarrow{i} \).

\[ y \]
\[ \overline{v} \]
\[ O \]
\[ -3 \]
\[ x \]

We define the unit vector \( \overrightarrow{j} = \langle 0, 1 \rangle \) as the direction of any vector whose initial point is \( O \) and whose terminal point is on the positive \( y \)-axis.
Exercise. Let \( \mathbf{u} = (0, 3) \). Show that \( \mathbf{u} = 3\mathbf{j} \).

Exercise. Let \( \mathbf{v} = (0, -3) \). Show that \( \mathbf{v} = -3\mathbf{j} \).

Theorem. Suppose \( \mathbf{u} = (u_1, u_2) \) is a vector. Then \( \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \).
Exercise. Prove the above theorem.

**Vector Addition Theorem.** Suppose $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors. Then $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.

Proof. By the Trichotomy Law, $u_1 > 0$, $u_1 = 0$, or $u_1 < 0$. The same is true for $u_2$, $v_1$, and $v_2$. If we look at all combinations of the above possibilities, there are $3^4 = 81$ cases to consider. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then the proof is obvious from the definition of the vector addition. Therefore, we will assume that both $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors. That still leaves 64 cases to consider. We will prove the theorem for the case $u_1 > 0$, $u_2 > 0$, $v_1 > 0$, and $v_2 > 0$. The methods and tools we use to prove this case can be used to prove all other cases.

If $\mathbf{u}$ and $\mathbf{v}$ have the same direction, then the proof is not difficult and will be left as an exercise. Therefore, assume that the directions of $\mathbf{u}$ and $\mathbf{v}$ are different.

Let $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OB}$. Let $\overrightarrow{OC}$ be the moved $\mathbf{v}$. Then $\mathbf{u} + \mathbf{v} = \overrightarrow{OC}$. We have to show that the coordinates of $C$ are $(u_1 + v_1, u_2 + v_2)$. (Without loss of generality, assume that $\mathbf{u}$ and $\mathbf{v}$ are the vectors as shown on the figure.)

Drop a perpendicular from $A$ to the $x$-axis and let the foot of this perpendicular be $D$. Drop a perpendicular from $B$ to the $x$-axis and let the foot of this perpendicular be $E$. Drop a perpendicular from $C$ to the $x$-axis and let the foot of this perpendicular be $F$. 
Drop a perpendicular from $A$ to the line $CF$ and let the foot of this perpendicular be $G$. Let $H$ be a point on the ray $\overrightarrow{OA}$ as shown in the figure.

$OB \parallel AC$ and $OA$ is a transversal. Therefore, $|\angle AOB| = |\angle HAC|$. (Corresponding angles.) Since both $OF$ and $AG$ are perpendicular to $CF$, $OF \parallel AG$. The line $OA$ is a transversal to the parallel lines $OF$ and $AG$. Therefore, $|\angle AOE| = |\angle HAG|$. Then $|\angle BOE| = |\angle CAG|$, because $|\angle BOE| = |\angle AOE| - |\angle AOB| = |\angle HAG| - |\angle HAC| = |\angle CAG|.$

By the ASA theorem, triangle $\triangle OBE$ is congruent to the triangle $\triangle ACG$. Therefore, $|AG| = |OE| = v_1$, and $|CG| = |BE| = v_2$.

Since $ADFG$ is a rectangle, $|GF| = |AD| = u_1$, and $|DF| = |AG| = v_1$.

Now, $|OF| = |OD| + |DF| = u_1 + v_1$ and $|CF| = |GF| + |CG| = u_2 + v_2$. Therefore, the coordinates of $C$ are $(u_1 + v_1, u_2 + v_2)$.

Scalar Multiplication Theorem. Suppose $\vec{u} = \langle u_1, u_2 \rangle$ is a vector and $k$ is a scalar. Then $k\vec{u} = \langle ku_1, ku_2 \rangle$.

Proof. By the Trichotomy Law, there are $3^3 = 27$ possible combinations of $k$, $u_1$ and $u_2$. If $k = 0$ or $\vec{u} = \vec{0}$, then proof follows immediately from the definition of the scalar multiplication. That eliminates 6 cases. We will consider the case $k > 0$, $u_1 > 0$, $u_2 > 0$. Tools and the methods used to prove the theorem in this case can be used to prove the theorem in all other cases.

Within this case, there are three cases. $k < 1$, $k > 1$ or $k = 1$. The proof of the case $k < 1$ is similar to the proof of the case $k > 1$. Therefore, we will prove the theorem for the following case. $k \geq 1$, $u_1 > 0$, and $u_2 > 0$.

If $k = 1$, then the proof follows immediately from the definition. That is, by the definition of the scalar multiplication,

$$1\vec{u} = \vec{u}$$

Therefore,

$$1\langle u_1, u_2 \rangle = \langle u_1, u_2 \rangle = \langle 1u_1, 1u_2 \rangle.$$
Now suppose $k > 1$.

Let $\overrightarrow{u} = \overrightarrow{OA}$ and $k\overrightarrow{u} = \overrightarrow{OB}$. We want to show that the coordinates of $B$ are $(ku_1, ku_2)$.

Drop a perpendicular from $A$ to the $x$-axis and let the foot of this perpendicular be $C$. Drop a perpendicular from $B$ to the $x$-axis and let the foot of this perpendicular be $D$. By the AA criterion, the triangles $\triangle OAC$ and $\triangle OBD$ are similar. Therefore,

$$\frac{|OD|}{|OC|} = \frac{|OB|}{|OA|}.$$ 

Therefore, by the cross-multiplication algorithm,

$$|OD| = |OC| \cdot \frac{|OB|}{|OA|} = u_1 \left( \frac{k\|\overrightarrow{u}\|}{\|\overrightarrow{u}\|} \right) = ku_1.$$ 

Also,

$$\frac{|BD|}{|AC|} = \frac{|OB|}{|OA|}.$$ 

Therefore, by the cross-multiplication algorithm,

$$|BD| = |AC| \cdot \frac{|OB|}{|OA|} = u_2 \left( \frac{k\|\overrightarrow{u}\|}{\|\overrightarrow{u}\|} \right) = ku_2.$$ 

Therefore, the coordinates of $B$ are $(ku_1, ku_2)$.

The following theorem follows easily from the Vector Addition Theorem and the Scalar Multiplication Theorem. The proof is left as an exercise.
Properties of Vectors Theorem. Suppose \( \overrightarrow{u} = \langle u_1, u_2 \rangle \), \( \overrightarrow{v} = \langle v_1, v_2 \rangle \), \( \overrightarrow{w} = \langle w_1, w_2 \rangle \) are vectors, \( k \), \( k_1 \), and \( k_2 \) are scalars. Then

1. \( \overrightarrow{u} + \overrightarrow{v} = \overrightarrow{v} + \overrightarrow{u} \)
2. \( \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{v} + \overrightarrow{u}) + \overrightarrow{w} \)
3. \( k_1(k_2\overrightarrow{u}) = (k_1k_2)\overrightarrow{u} \)
4. \( 0\overrightarrow{u} = \overrightarrow{0} \)
5. \( \overrightarrow{0} + \overrightarrow{u} = \overrightarrow{u} \)

Exercise. Prove the Properties of Vectors Theorem.

16.4 Dot Product Between Two Vectors

Let \( \overrightarrow{u} = \langle u_1, u_2 \rangle \) and \( \overrightarrow{v} = \langle v_1, v_2 \rangle \) be two vectors. We define the dot product between \( \overrightarrow{u} \) and \( \overrightarrow{v} \), denoted by \( \overrightarrow{u} \cdot \overrightarrow{v} \), as follows.

\[
\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2.
\]

The above definition is purely an algebraic definition. Does it have a geometrical meaning? To find out we will investigate the information we can obtain from the above definition.

16.4.1 Properties of the Dot Product

(P1) By definition, the dot product between two given vectors is a scalar. If one of the two vectors is the zero vector, then the dot product is 0.

(P2) The dot product is commutative. That is,

\[
\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}
\]

Proof.

\[
\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2, \text{ by definition.}
\]
\[
\overrightarrow{v} \cdot \overrightarrow{u} = v_1u_1 + v_2u_2, \text{ also by definition.}
\]
\[
\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}, \text{ since numbers commute.}
\]
(P3) Following the same line of thinking and using the Vector Addition Theorem, you can prove the following vector identity.
Suppose \( \mathbf{u} = \langle u_1, u_2 \rangle \), \( \mathbf{v} = \langle v_1, v_2 \rangle \), and \( \mathbf{w} = \langle w_1, w_2 \rangle \). Then

\[
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
\]

(P4) Following the same line of thinking and using the Scalar Multiplication Theorem, you can prove the following vector identity.
Suppose \( \mathbf{u} = \langle u_1, u_2 \rangle \), \( \mathbf{v} = \langle v_1, v_2 \rangle \), and \( k \) is a scalar. Then

\[
k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v}).
\]

(P5) If \( \mathbf{u} = \langle u_1, u_2 \rangle \), then

\[
|\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2.
\]

Proof.

\[
\mathbf{u} \cdot \mathbf{u} = u_1 u_1 + u_2 u_2, \text{ by definition.}
\]

\[
= u_1^2 + u_2^2
\]

\[
= \left(\sqrt{u_1^2 + u_2^2}\right)^2
\]

\[
= \|\mathbf{u}\|^2.
\]

16.4.2 Angle Between Two Non-Zero Vectors

Consider two non-zero vectors \( \mathbf{u} \) and \( \mathbf{v} \). Move the two vectors so that the initial point of both vectors coincide. Let the initial point of the two vectors be \( O \), the terminal point of \( \mathbf{u} \) be \( A \), and the terminal point of \( \mathbf{v} \) be \( B \).
There are two angles with the initial side $\overrightarrow{OA}$ and the terminal side $\overrightarrow{OB}$ with measure in $[0, 2\pi]$. The *smaller* angle $\theta$ of the two is called the *angle between* $\overrightarrow{u}$ and $\overrightarrow{v}$.

If $\theta$ is the angle between two non-zero vectors $\overrightarrow{u}$ and $\overrightarrow{v}$, then by definition,

$$0 \leq \theta \leq \pi.$$ 

When $\overrightarrow{u}$ and $\overrightarrow{v}$ have the same direction, then $\theta = 0$, and when $\overrightarrow{u}$ and $\overrightarrow{v}$ have the opposite direction, then $\theta = \pi$.

We say two non-zero vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are *orthogonal* if the angle $\theta$ between them is a right angle.

**The Dot Product Theorem.** Let $\overrightarrow{u}$ and $\overrightarrow{v}$ be two non-zero vectors so that the angle between them is $\theta$. Then

$$\overrightarrow{u} \cdot \overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\| \cos \theta.$$

*Proof.* Move the two vectors so that the initial points of both vectors coincide. Let the initial point of the two vectors be $O$, the terminal point of $\overrightarrow{u}$ be $A$, and the terminal point of $\overrightarrow{v}$ be $B$. Let $\overrightarrow{AB}$ be $\overrightarrow{w}$. Then by the definition of the vector addition, $\overrightarrow{w} = \overrightarrow{v} - \overrightarrow{u}$.
By the Law of Cosines Theorem,
\[ \| \mathbf{v} - \mathbf{u} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 - 2 \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta. \]

The left side of the above equation = \( \| \mathbf{v} - \mathbf{u} \|^2 \)
\[ = (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}), \text{ by P5.} \]
\[ = (\mathbf{v} + (\mathbf{u})) \cdot (\mathbf{v} + (\mathbf{u})) \]
\[ = (\mathbf{v} + (\mathbf{u})) \cdot \mathbf{v} + (\mathbf{v} + (\mathbf{u})) \cdot (\mathbf{u}), \text{ by P3.} \]
\[ = \mathbf{v} \cdot (\mathbf{v} + (\mathbf{u})) + (\mathbf{u}) \cdot (\mathbf{v} + (\mathbf{u})), \text{ by P2.} \]
\[ = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot (\mathbf{u}) + (\mathbf{u}) \cdot \mathbf{v} + (\mathbf{u}) \cdot (\mathbf{u}), \text{ by P3.} \]
\[ = \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}, \text{ by P4.} \]
\[ = \mathbf{v} \cdot \mathbf{v} - 2 \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}, \text{ by P2.} \]
\[ = \| \mathbf{v} \|^2 - 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2, \text{ by P5.} \]

Therefore,
\[ \| \mathbf{v} \|^2 - 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{u} \|^2 = \| \mathbf{v} \|^2 + \| \mathbf{v} \|^2 - 2 \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta. \]

Collecting like terms,
\[ -2 \mathbf{u} \cdot \mathbf{v} = -2 \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta. \]

Multiplying both sides by \(-\frac{1}{2}\), we get:
\[ \mathbf{u} \cdot \mathbf{v} = \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta. \]

Exercise. Find the angle between \( \mathbf{u} = \langle 3, 4 \rangle \) and \( \mathbf{v} = \langle 1, 7 \rangle \). Round your answer to the nearest degree.
Let \( \theta \) be the angle between \( \overline{u} \) and \( \overline{v} \).

\[
\overline{u} \cdot \overline{v} = (3)(1) + (4)(7) = 31.
\]

\[
\|\overline{u}\| = \sqrt{3^2 + 4^2} = 5.
\]

and

\[
\|\overline{v}\| = \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2}.
\]

Therefore, by the Dot Product Theorem,

\[
31 = (5)(5\sqrt{2}) \cos \theta.
\]

That is,

\[
\cos \theta = \frac{31}{25\sqrt{2}}.
\]

and

\[
\theta = \cos^{-1} \left( \frac{31}{25\sqrt{2}} \right) \approx 29^\circ. \quad \square
\]

Since \( \overline{u} \cdot \overline{v} \) is a scalar, (see P1), the Trichotomy Law holds.

**Corollary to the Dot Product Theorem.** Let \( \overline{u} \) and \( \overline{v} \) be two non-zero vectors so that the angle between them is \( \theta \). Then

1. \( \overline{u} \cdot \overline{v} = 0 \) if and only if \( \overline{u} \) and \( \overline{v} \) are orthogonal.
2. \( \overline{u} \cdot \overline{v} > 0 \) if and only if \( \theta \) is acute.
3. \( \overline{u} \cdot \overline{v} < 0 \) if and only if \( \theta \) is obtuse.

**Proof of (1).** Suppose \( \overline{u} \cdot \overline{v} = 0 \).

Then by the Dot Product Theorem, \( 0 = \|\overline{u}\| \|\overline{v}\| \cos \theta \). Since \( \overline{u} \) and \( \overline{v} \) are two non-zero vectors, \( \|u\| \neq 0 \) and \( \|v\| \neq 0 \). Therefore, \( \cos \theta = 0 \). The general solution of this equation is \( \theta = 2n\pi \pm \cos^{-1}(0) \), for any integer \( n \). By definition, \( \theta \) lies in the interval \( [0, \pi] \). The only particular solution of \( \cos \theta = 0 \) in the interval \( [0, \pi] \) is \( \cos^{-1}(0) \). That is, \( \theta = \frac{\pi}{2} \).

Now suppose \( \overline{u} \) and \( \overline{v} \) are orthogonal.

Then \( \cos \theta = 0 \). By the Dot Product Theorem, \( \overline{u} \cdot \overline{v} = 0 \). \( \square \)
16.4. DOT PRODUCT BETWEEN TWO VECTORS

Proof of (2). Suppose \( \bar{u} \cdot \bar{v} > 0 \).

Then by the Dot Product Theorem, \( 0 < \|\bar{u}\|\|\bar{v}\| \cos \theta \). Since \( \bar{u} \) and \( \bar{v} \) are two non-zero vectors, \( \|u\| > 0 \) and \( \|v\| > 0 \). Therefore, \( \cos \theta > 0 \). By definition, \( \theta \) lies in the interval \([0, \pi]\). Since \( \cos \theta > 0 \), \( \theta \) lies in the first quadrant. Therefore, \( \theta \) is acute.

Now suppose \( \theta \) is acute.

Then \( \cos \theta > 0 \). Since \( \bar{u} \) and \( \bar{v} \) be two non-zero vectors, \( \|u\| > 0 \) and \( \|v\| > 0 \). Therefore, \( \|\bar{u}\|\|\bar{v}\| \cos \theta > 0 \). By the Dot Product Theorem, \( \bar{u} \cdot \bar{v} > 0 \).

The proof of (3) is left as an exercise.

Exercise. Prove part (3) of the Corollary to the Dot Product Theorem.

16.4.3 Projection of a Vector Along Another Vector

We have seen that any non-zero vector \( \bar{u} = \langle u_1, u_2 \rangle \) can be written as a sum of two orthogonal vectors; namely \( u_1\bar{i} \) and \( u_2\bar{j} \). The vector \( u_1\bar{i} \) is called the projection of \( \bar{u} \) in the direction of \( \bar{i} \). The vector \( u_2\bar{j} \) is called the projection of \( \bar{u} \) orthogonal to \( \bar{i} \).

Can we write a given non-zero vector \( \bar{u} = \langle u_1, u_2 \rangle \) as a sum of two orthogonal vectors so that one of the orthogonal vectors is along a given vector \( \bar{a} \)? The answer is yes.

Suppose \( \bar{u} \) and \( \bar{a} \) are non-zero vectors and they have a common initial point. Suppose the angle between them is \( \theta \). We will denote the vector the projection of \( \bar{u} \) in the direction of \( \bar{a} \) as \( \text{proj}_{\bar{a}} \bar{u} \).

The angle \( \theta \) can be acute, obtuse, right, or zero.

If \( \theta = 0 \), then clearly the \( \text{proj}_{\bar{a}} \bar{u} = \bar{u} \) and the vector the projection of \( \bar{u} \) orthogonal to \( \bar{a} \) is \( \bar{0} \). If \( \theta \) is right, then the \( \text{proj}_{\bar{a}} \bar{u} = \bar{0} \) and the vector the projection of \( \bar{u} \) orthogonal to \( \bar{a} \) is \( \bar{u} \).

Now let us consider the other two cases.
In the following figure, the “blue” vector is the vector \( \text{proj}_a \mathbf{u} \), and the “red” vector is the projection of \( \mathbf{u} \) orthogonal to \( a \).

When \( \theta \) is acute, the norm of \( \text{proj}_a \mathbf{u} \) is \( \| \mathbf{u} \| \cos \theta \) and the direction of \( \text{proj}_a \mathbf{u} \) is \( \frac{1}{\| a \|} a \).

When \( \theta \) is obtuse, the norm of \( \text{proj}_a \mathbf{u} \) is \( -\| \mathbf{u} \| \cos \theta \) and the direction of \( \text{proj}_a \mathbf{u} \) is \( -\frac{1}{\| a \|} a \).

Therefore, in either case,

\[
\text{proj}_a \mathbf{u} = \| \mathbf{u} \| \cos \theta \frac{1}{\| a \|} a.
\]

If we scalar multiply the right side of above equation by \( \frac{\| a \|}{\| a \|} \), then

\[
\text{proj}_a \mathbf{u} = \| \mathbf{u} \| \| a \| \cos \theta \frac{1}{\| a \|^2} a.
\]

Now by the Dot Product Theorem,

\[
\text{proj}_a \mathbf{u} = \frac{\mathbf{u} \cdot a}{\| a \|^2} a.
\]

By the definition of vector addition,

\[
\text{the projection of } \mathbf{u} \text{ orthogonal to } a = \mathbf{u} - \text{proj}_a \mathbf{u}.
\]

Notice that we can write \( \text{proj}_a \mathbf{u} \) as:

\[
\text{proj}_a \mathbf{u} = \left( \frac{\mathbf{u} \cdot a}{\| a \|} \right) \frac{1}{\| a \|} a.
\]
Therefore,
\[ \|\text{proj}_\mathbf{v}\mathbf{u}\| = \left| \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|} \right|. \]

Since \( \|\mathbf{a}\| > 0 \), the above is the same as
\[ \|\text{proj}_\mathbf{v}\mathbf{u}\| = \left| \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|} \right|. \]

**Example.** Suppose \( \mathbf{u} = \langle 3, 4 \rangle \) and \( \mathbf{v} = \langle 1, 7 \rangle \). Find \( \text{proj}_\mathbf{v}\mathbf{u} \) and \( \|\text{proj}_\mathbf{v}\mathbf{u}\| \).

**Answer.**
\[ \mathbf{u} \cdot \mathbf{v} = (3)(1) + (4)(7) = 31. \]
\[ \|\mathbf{v}\| = \sqrt{1^2 + 7^2} = \sqrt{50}. \]

Therefore,
\[ \text{proj}_\mathbf{v}\mathbf{u} = \frac{31}{\sqrt{50}} \langle 1, 7 \rangle. \]

and
\[ \|\text{proj}_\mathbf{v}\mathbf{u}\| = \frac{31}{\sqrt{50}}. \]

16.4.4 Work Done by a Constant Force moving an Object along a Straight Line

Consider a little boy pulling a cart along the \( x \)-axis in the direction of \( \vec{i} \) with a constant force \( \overrightarrow{F} \) (in force units).
Suppose the cart moves along the $x$-axis a distance of $d$ (in distance units). The vector $\vec{d} = d\hat{i}$ is called the \textit{displacement vector}. The component of $\vec{F}$ that contributes to the motion of the cart is $\text{proj}_i\vec{F}$. The \textit{work} done by the boy (by the constant force $\vec{F}$) moving the cart a distance $d$ is defined as:

The work done by a constant force $\vec{F}$ moving an object a distance $d = \|\text{proj}_i\vec{F}\| \cdot d$.

That is,

\[
\text{work done} = \left( \frac{\vec{F} \cdot \hat{i}}{\|\hat{i}\|} \right) \cdot d
\]

\[
= (\vec{F} \cdot \hat{i})d, \text{ because } \hat{i} \text{ is a unit vector.}
\]

\[
= \vec{F} \cdot (d\hat{i}), \text{ by P4.}
\]

\[
= \vec{F} \cdot \vec{d}
\]

\[\textbf{Work Done by a Constant Force Theorem.} \] Suppose an object moves along a straight line by the influence of a constant force $\vec{F}$. Suppose $\vec{d}$ is the displacement vector. Then the work done by $\vec{F}$ moving the object a distance $d$ is $\vec{F} \cdot \vec{d}$. 

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By M. Sunil R. Koswatta. First draft initiated: August 1, 2014.
First draft completed: December 31, 2014

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